Theorem 6.71.1 (continued 1)

\[ \int_C f(z) \, dz = \int_{-C_0} f(z) \, dz = - \int_{C_0} f(z) \, dz. \]
Then, by the definition of residue at infinity, \( \int_C f(z) \, dz = \text{Res}_{z=\infty} f(z) \). Now we take the Laurent series of \( f \) about \( z_0 = 0 \) (\( f \) may or may not be analytic at \( z_0 \)) to get by Theorem 60.1, “Laurent’s Theorem,”
\[ f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \]
for \( R_1 < |z| < \infty \) where \( R_1 < R_0 \) is such that \( C \subset \{ z \mid |z| < R_1 \} \) and
\[ c_n = \frac{1}{2\pi i} \int_{C_0} \frac{f(z) \, dz}{z^{n+1}} \]
for \( n \in \mathbb{Z} \) (see the note after Theorem 60.1 for the concise expression of \( c_n \)). Replacing \( z \) with \( 1/z \) in the Laurent series for \( f(z) \) and then multiplying by \( 1/z^2 \) gives
\[ \frac{1}{z^2} f \left( \frac{1}{z} \right) = \frac{1}{z^2} \sum_{n=-\infty}^{\infty} c_n z^n = \sum_{n=-\infty}^{\infty} c_n z^{n+2} \]
for \( 0 < |z| < \frac{1}{R_1} \).

Theorem 6.71.1 (continued 2)

Proof (continued). With \( n = 1 \) we get
\[ \text{Res}_{z=0} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right) = c_{-1} = \frac{1}{2\pi i} \int_{C_0} f(z) \, dz. \]
Now by the definition of \( \text{Res}_{z=\infty} f(z) \),
\[ \text{Res}_{z=\infty} f(z) = \frac{1}{2\pi i} \int_C f(z) \, dz = \frac{1}{2\pi i} \int_{C_0} f(z) \, dz = \text{Res}_{z=0} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right) \]
and
\[ \int_C f(z) \, dz = 2\pi i \text{Res} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right), \]
as claimed.