Complex Variables

Chapter 6. Residues and Poles
Section 6.7.1. Residues at Infinity—Proofs of Theorems
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$$\int_C f(z) \, dz = 2\pi i \text{Res}_{z=0} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right).$$

Proof. Choose $R_0 > 0$ sufficiently large so that $C \subset \{ z \mid |z| < R_0 \}$ and define $C_0$ as the circle $|z| = R_0$ oriented in the negative direction. See Figure 89.
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**Proof.** Choose $R_0 > 0$ sufficiently large so that $C \subset \{ z \mid |z| < R_0 \}$ and define $C_0$ as the circle $|z| = R_0$ oriented in the negative direction. See Figure 89.
\[ \int_C f(z) \, dz = \int_{-C_0} f(z) \, dz = -\int_{C_0} f(z) \, dz. \]
Then, by the definition of residue at infinity, \( \int_C f(z) \, dz = \text{Res}_{z=\infty} f(z). \)
Now we take the Laurent series of \( f \) about \( z_0 = 0 \) (\( f \) may or may not be analytic at \( z_0 \)) to get by Theorem 60.1, “Laurent’s Theorem,”
\[ f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \]
for \( R_1 < |z| < \infty \) where \( R_1 < R_0 \) is such that \( C \subset \{ z \mid |z| < R_1 \} \) and
\[ c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z) \, dz}{z^{n+1}} \]
for \( n \in \mathbb{Z} \) (see the note after Theorem 60.1 for the concise expression of \( c_n \)).
Theorem 6.71.1 (continued 1)

**Proof (continued).** By Theorem 4.49.B, “Principle of Deformation,”
\[ \int_C f(z) \, dz = \int_{-C_0} f(z) \, dz = -\int_{C_0} f(z) \, dz. \]
Then, by the definition of residue at infinity,
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for \( n \in \mathbb{Z} \) (see the note after Theorem 60.1 for the concise expression of \( c_n \)).
Replacing \( z \) with \( 1/z \) in the Laurent series for \( f(z) \) and then multiplying by \( 1/z^2 \) gives
\[
\frac{1}{z^2} f \left( \frac{1}{z} \right) = \frac{1}{z^2} \sum_{n=-\infty}^{\infty} \frac{c_n}{z^n} = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} \quad \text{for} \quad 0 < |z| < \frac{1}{R_1}.
\]
\[ \int_{C} f(z) \, dz = \int_{-C_0} f(z) \, dz = -\int_{C_0} f(z) \, dz. \]
Then, by the definition of residue at infinity, \( \int_{C} f(z) \, dz = \text{Res}_{z=\infty} f(z). \) Now we take the Laurent series of \( f \) about \( z_0 = 0 \) (\( f \) may or may not be analytic at \( z_0 \)) to get by Theorem 60.1, “Laurent’s Theorem,” \( f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \) for \( R_1 < |z| < \infty \) where \( R_1 < R_0 \) is such that \( C \subset \{ z \mid |z| < R_1 \} \) and
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c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z) \, dz}{z^{n+1}} \quad \text{for } n \in \mathbb{Z} \] (see the note after Theorem 60.1 for the concise expression of \( c_n \)). Replacing \( z \) with \( 1/z \) in the Laurent series for \( f(z) \) and then multiplying by \( 1/z^2 \) gives
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\]
Theorem 6.71.1 (continued 2)

Proof (continued). With $n = 1$ we get

$$\text{Res}_{z=0} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right) = c_{-1} = \frac{1}{2\pi i} \int_{-C_0} f(z) \, dz.$$ 

Now by the definition of $\text{Res}_{z=\infty} f(z)$,

$$\text{Res}_{z=\infty} f(z) = \frac{1}{2\pi i} \int_{C} f(z) \, dz = \frac{1}{2\pi i} \int_{-C_0} f(z) \, dz = \text{Res}_{z=0} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right)$$

and

$$\int_{C} f(z) \, dz = 2\pi i \text{Res} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right),$$

as claimed.
Theorem 6.71.1 (continued 2)

Proof (continued). With $n = 1$ we get

$$\text{Res}_{z=0} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right) = c_{-1} = \frac{1}{2\pi i} \int_{-C_0} f(z) \, dz.$$

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