Theorem 6.76.1

Theorem 6.76.1. Suppose that

(a) two functions $p$ and $q$ are analytic at a point $z_0$, and
(b) $p(z_0) \neq 0$ and $q$ has a zero of order $m$ at $z_0$.

Then the quotient $p(z)/q(z)$ has a pole of order $m$ at $z_0$.

**Proof.** Since $q$ has a zero of order $m$ at $z_0$ then by Theorem 6.75.2, there is an isolated singularity at $z_0$ throughout which $q(z) \neq 0$. So $p/q$ has an isolated singularity at $z_0$ and by Theorem 6.75.1, we have $q(z) = (z-z_0)^mg(z)$ where $g$ is analytic and nonzero at $z_0$. So

$$
\frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)^m g(z)} = \frac{p(z)/g(z)}{(z-z_0)^m} = \frac{\varphi(z)}{(z-z_0)^m}
$$

where $\varphi(z) = p(z)/q(z)$ is analytic and nonzero (by hypothesis (b)) are $z_0$. So by Theorem 6.73.1, $z_0$ is a pole of order $m$ of $p/q$, as claimed. \(\square\)

Theorem 6.76.2

**Theorem 6.76.2.** Let the functions $p$ and $q$ be analytic at $z_0$. If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) = 0$ (that is, $q$ has a zero of multiplicity one at $z_0$) then $z_0$ is a simple pole of $p/q$ and $\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.

**Proof.** By Theorem 6.75.1, $q(z) = (z-z_0)g(z)$ where $g$ is analytic and nonzero at $z_0$. So by Theorem 6.76.1, $p/q$ has a simple pole at $z_0$. So, as seen in the proof of Theorem 6.76.1, $\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{z-z_0} = \frac{\varphi(z)}{z-z_0}$. So by Theorem 6.73.1, $\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \varphi(z_0) = \frac{p(z_0)}{g(z_0)}$. But since $g(z) = (z-z_0)g(z)$ then $q'(z) = [1]g(z) + (z-z_0)[g'(z)]$ and $q'(z_0) = g(z_0)$, so

$$
\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)},
$$

as claimed. \(\square\)