**Theorem 6.77.1**

**Theorem 6.77.1.** If \( z_0 \) is a pole of a function \( f \) then \( \lim_{z \to z_0} f(z) = \infty \).

**Proof.** Suppose \( f \) has a pole of order \( m \) at \( z = z_0 \). Then by Theorem 6.73.1, \( f(z) = \frac{\varphi(z)}{(z-z_0)^m} \) where \( \varphi \) is analytic for \( |z-z_0| < R_2 \) for some \( R_2 > 0 \) and \( \varphi(z_0) \neq 0 \). Then

\[
\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z-z_0)^m}{\varphi(z)} = \lim_{z \to z_0} \frac{z-z_0}{\varphi(z)} = 0, \quad \varphi(z_0) \neq 0.
\]

So \( \lim_{z \to z_0} f(z) = \infty \) by Theorem 2.17.1. \( \square \)

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**Theorem 6.77.2**

**Theorem 6.77.2.** If \( z_0 \) is a removable singular point of a function \( f \), then \( f \) is analytic and bounded in some deleted neighborhood \( 0 < |z-z_0| < \varepsilon \) of \( z_0 \).

**Proof.** Since a removable singular point is isolated (by definition) then \( f \) is analytic for \( 0 < |z-z_0| < R_2 \) for some \( R_2 > 0 \). By Note 6.72.A, there is analytic \( g \) defined for \( |z-z_0| < R_2 \) such that \( g(z) = f(z) \) for \( 0 < |z-z_0| < R_2 \). Let \( \varepsilon > 0 \) satisfy \( \varepsilon < R_2 \). Then \( g \) is continuous on \( |z-z_0| \leq \varepsilon \) and so by Theorem 2.18.3 there is \( M \) such that \( |g(z)| \leq M \) for all \( |z-z_0| \leq \varepsilon \). Therefore \( |f(z)| \leq M \) for all \( 0 < |z-z_0| \leq \varepsilon \) and the claim holds. \( \square \)

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**Lemma 6.77.1.** Riemann’s Theorem

**Lemma 7.77.1.** Riemann’s Theorem.

Suppose that a function \( f \) is analytic and bounded in some deleted neighborhood \( 0 < |z-z_0| < \varepsilon \) of \( z_0 \). If \( f \) is not analytic at \( z_0 \), then \( f \) has a removable singularity at \( z_0 \).

**Proof.** Since \( f \) is analytic in \( 0 < |z-z_0| < \varepsilon \) then by Theorem 60.1, “Laurent’s Theorem,” there is a Laurent series for \( f \) centered at \( z_0 \):

\[
f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{for} \quad 0 < |z-z_0| < \varepsilon.
\]

Let \( C \) denote the positively oriented circle \( |z-z_0| = \rho \) where \( 0 < \rho < \varepsilon \) (so that \( f \) is analytic on \( C \)). By Laurent’s Theorem, \( b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \) for \( n = 1, 2, \ldots \).
Lemma 6.77.1. Riemann’s Theorem (continued)

Lemma 7.77.1. Riemann’s Theorem.
Suppose that a function \( f \) is analytic and bounded in some deleted neighborhood \( 0 < |z - z_0| < \varepsilon \) of \( z_0 \). If \( f \) is not analytic at \( z_0 \), then \( f \) has a removable singularity at \( z_0 \).

Proof (continued). Since \( f \) is hypothesized to be bounded on \( 0 < |z - z_0| < \varepsilon \), let \( M \) be the bound and then

\[
|b_n| = \left| \frac{1}{2\pi i} \int \frac{f(z) \, dz}{(z - z_0)^{n+1}} \right| \\
\leq \frac{1}{2\pi} M 2\pi \varepsilon^{n+1} = M \varepsilon^n
\]

by Theorem 4.43.1.

Since \( 0 < \rho < \varepsilon \) is arbitrary, this inequality holds for all such \( \rho \) and hence

\[
|b_n| = \lim_{\rho \to 0} |b_n| \leq \lim_{\rho \to 0} M \rho^n = 0.
\]

That is, \( b_n = c_n = 0 \) for all \( n = 1, 2, \ldots \) and the Laurent series for \( f \) is \( f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \). So, by definition, the singular point \( z_0 \) of \( f \) is a removable singular point.

Theorem 6.73.1. Cauchy-Weierstrass Theorem

Suppose that \( z_0 \) is an essential singularity of function \( f \) and let \( w_0 \) be any complex number. Then for all \( \varepsilon > 0 \), the inequality \( |f(z) - w_0| < \varepsilon \) is satisfied at some point \( z \) in every deleted neighborhood \( 0 < |z - z_0| < \delta \) of \( z_0 \) for \( \delta > 0 \).

Proof. Let \( w_0 \in \mathbb{C}, \varepsilon > 0, \) and \( \delta > 0 \) be given where \( \delta \) is sufficiently small so that \( f \) is analytic on \( 0 < |z - z_0| < \delta \). Assume \( |f(z) - w_0| \geq \varepsilon \) for all \( z \) in \( 0 < |z - z_0| < \delta \). Then the function \( g(z) = 1/(f(z) - w_0) \) is analytic and bounded (by \( M = 1/\varepsilon \)) on \( 0 < |z - z_0| < \delta \) (notice that \( g \) is non-zero by the definition for these \( z \) values). So by Lemma 6.77.1, \( z_0 \) is a removable singularity of \( g \). We extend \( g \) to be defined at \( z_0 \) by setting \( g(z_0) = \lim_{z \to z_0} g(z) \). Then \( g \) is analytic on \( |z - z_0| < \delta \) (see Note 6.72.A).

Theorem 6.73.1 (continued)

Proof (continued). If \( g(z_0) \neq 0 \) then \( f(z) = \frac{1}{g(z)} + w_0 \) and \( f \) is analytic where \( g \) is non-zero. Since \( g(z_0) \neq 0 \) then \( g(z) \neq 0 \) for \( |z - z_0| < \delta \). But then \( f \) is analytic on \( 0 < |z - z_0| < \delta \) and \( \lim_{z \to z_0} f(z) = \frac{1}{g(z_0)} + w_0 \). So from the definition of limit, there is \( \delta_1 \) such that \( 0 < \delta_1 < \delta \) and \( f \) is bounded on \( 0 < |z - z_0| < \delta_1 \). But then, by Lemma 6.73.1, \( f \) has a removable singular point at \( z = z_0 \), not an essential singularity, a CONTRADICTION.

If \( g(z_0) = 0 \) then, since \( g \) is not identically the zero function (since \( g \) is nonzero for \( 0 < |z - z_0| < \delta \)) then \( z_0 \) is a zero of \( g \) of some order \( m \) (see Section 75) and so by Theorem 6.76.1 (with \( p(z) = 1 + g(z)w_0 \) and \( q(z) = g(z) \)), \( f(z) = \frac{1}{g(z)} + w_0 = \frac{1 + g(z)w_0}{g(z)} \) has a pole of order \( m \) at \( z_0 \). CONTRADICTING the fact that \( f \) has an essential singularity, no a pole at \( z_0 \). So the assumption that \( |f(z) - w_0| \geq \varepsilon \) for all \( 0 < |z - z_0| < \delta \) is false and so there must be some point \( z \) in \( 0 < |z - z_0| < \delta \) such that all \( 0 < |z - z_0| < \delta \) is false and so there must be some point \( z \) in \( 0 < |z - z_0| < \delta \) such that \( |f(z) - w_0| < \varepsilon \), as claimed.