# Complex Variables, MATH 4337/5337, Fall 2020

Homework 6, 2.17. Limits Involving the Point at Infinity, 2.18. Continuity

Due Tuesday, March 3 at 12:45

**Write in complete sentences!!!** *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. The exercise numbers are based on the 9th edition of the textbook.

**2.18.10.** Use Theorem 2.17.1 to show the following.

**(a)** $\lim_{z\to\infty} \dfrac{4z^2}{(z-1)^2} = 4.$

**(b)** $\lim_{z\to 1} \dfrac{1}{(z-1)^3} = \infty.$

**(c)** $\lim_{z\to\infty} \dfrac{z^2+1}{z-1} = \infty.$

**2.18.13. (Graduate)** Prove that a set $S \subset \mathbb{C}$ is unbounded if and only if every neighborhood of the point at infinity contains at least one point in $S$. HINT: For the "only if" part use the definition of unbounded. For the "if" part, give a proof by contradiction.

**2.20.3.** Use the properties of derivatives given in Section 2.20 to show the following.

**(a)** A polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ (where $a_n \neq 0$) of degree $n$ (where $n \geq 1$) is differentiable everywhere with derivative $P'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots + na_n z^{n-1}$.

**(a′) (Bonus)** A polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = \sum_{j=0}^n a_k z^j$ (where $a_n \neq 0$) of degree $n$ (where $n \geq 1$) is $k$ time differentiable everywhere, where $0 \leq k \leq n$ (we interpret $P^{(0)}(z)$ as $P(z)$) with $k$th derivative

$$P^{(k)}(z) = \sum_{j=0}^{n-k} \frac{(k+j)!}{j!} a_{k+j} z^j.$$

Notice that $P^{(k)}(z)$ is an $n-k$ degree polynomial. Use mathematical induction.

**(b)** Show that coefficients in the polynomial $P(z)$ in part (a) can be written

$$a_0 = P(0),\ a_1 = \frac{P'(0)}{1!},\ a_2 = \frac{P''(0)}{2!}, \ldots, a_n = \frac{P^{(n)}(0)}{n!}.$$

**2.20.4.** Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use the definition of derivative from Section 2.19 to prove that $\lim\limits_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{f'(z_0)}{g'(z_0)}$. Notice that this is L'Hopital's Rule in the complex setting.

**2.20.6.** Prove that for $n \in \mathbb{N}$, $\frac{d}{dz}[z^n] = nz^{n-1}$ using the technique described.

**(b)** Using the definition of derivative and the Binomial Theorem (Theorem 1.3.2).