Complex Variables, MATH 4337/5337, Spring 2020

Homework 7, 2.26. Harmonic Functions, 2.27. Uniquely Determined Analytic Functions, 2.28. Reflection Principle

Due Thursday, April 2 at 12:45

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. The exercise numbers are based on the 9th edition of the textbook.

2.27.1. Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain D that does not include the origin. Using the Cauchy-Riemann equations in polar coordinates and assuming continuity of all partial derivatives, prove that throughout D the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta} = 0,$$

which is the *polar form of Laplace's equation*. HINT: Use Theorem 2.21.A and Lemma 2.23.A and differentiate the Cauchy-Riemann equations in polar coordinates to produce second order partial derivatives.

9.115.1. Show that $u(x, y)$ is a harmonic function in some domain and follow the process in Example 2.26.5 in the notes to find a harmonic conjugate $v(x, y)$.

(a) $u(x, y) = 2x - x^3 + 3xy^2$. HINT: Use Theorem 2.22.A and Theorem 2.26.2. The answer is $v(x, y) = 2y - 3x^2y + y^3 + C$.

9.115.3. (Graduate) Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Prove that $u(x, y)$ and $v(x, y)$ must be constant throughout D. HINT: Use Theorem 2.26.2 to explain why Theorem 2.21.A applies.

2.29.1. Use Theorem 2.27.A to prove that if $f(z)$ is analytic and not constant throughout a domain D, then it cannot be constant throughout any neighborhood lying in D. HINT: ASSUME that $f(z) = w_0$ for some constant $w_0 \in \mathbb{C}$ on some neighborhood N lying in D and derive a CONTRADICTION.

2.29.4. We know from Example 2.22.1 that the function $f(z) = f(x+iy) = e^x e^{iy} = e^x \cos y + ie^x \sin y$ has a derivative everywhere in $\mathbb{C}$. Point out how it follows from the Reflection Principle (Theorem 2.28.A) that $\overline{f(z)} = f(\overline{z})$ for each $z \in \mathbb{C}$. Then verify this directly.