Complex Variables, MATH 4337/5337, Fall 2024

Homework 10: Section 31. Branches and Derivatives of Logarithms, Section 32. Some Identities Involving Logarithms

Due Saturday, April 20 at 11:59 pm

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. The exercise numbers are based on the 9th edition of the textbook.

3.36.5. Show that the principal nth root of nonzero complex number z_0 as defined in Section 1.9 (as $c_0 = \sqrt[n]{r}\exp(i\theta_0/n)$ where θ_0 is the principal argument of z_0) is the same as the principal value of $z_0^{1/n}$ defined in this section (as P.V. $z^{1/n} = e^{(1/n)\mathrm{Log}\, z}$).

3.36.8. Let c, c_1, c_2, and z denote complex numbers, where $z \neq 0$. Prove that if all the powers involved are principal values, then the following hold.

(a) $z^{c_1} z^{c_2} = z^{c_1+c_2}$.

(b) $z^{c_1}/z^{c_2} = z^{c_1-c_2}$.

(c) $(z^c)^n = z^{cn}$ where $n \in \mathbb{N}$.

3.38.3 and 4a. Exercise 3.38.2(b) implies $\sin(z+z_2) = \sin z \cos z_2 + \cos z \sin z_2$. By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

3.38.4. Show that for $z \in \mathbb{C}$, $\cos^2 z + \sin^2 z = 1$ as follows.

(a) Using Exercise 3.38.3 and the relations $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$.

3.38.2. (Graduate) (a) With the aid of the relationship

$$\cos z + i \sin z = \frac{e^{iz}+e^{-iz}}{2} + i\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} + \frac{e^{iz}-e^{-iz}}{2} = e^{iz}$$

show that

$$e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use the relations $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$ to show that

$$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

(b) Use the results in part (a) and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}) = \frac{1}{2i}(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2})$$

to obtain the identity $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.