Complex Variables, MATH 4337/5337, Fall 2024

Homework 7: Section 19. Derivatives, Section 21. Cauchy-Riemann Equations,

Section 22. Sufficient Conditions for Differentiability

Due Saturday, March 30 at 11:59 pm

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. The exercise numbers are based on the 9th edition of the textbook.

2.20.4. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use the definition of derivative from Section 2.19 to prove that $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{f'(z_0)}{g'(z_0)}$. Notice that this is L'Hopital's Rule in the complex setting.

2.20.8. Use the technique of Example 2.19.2 to show that $f'(z)$ does not exist at any point of $\mathbb{C}$ for:

(a) $f(z) = \mathrm{Re}(z)$.

2.24.1. Use Theorem 2.21.A to show that $f'(z)$ does not exist at any point.

(a) $f(z) = \overline{z}$.

(c) $f(z) = 2x + ixy^2$.

2.24.2. Use Theorem 2.22.A to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere and find $f''(z)$.

(b) $f(z) = e^{-x}e^{-iy}$.

2.24.8. (Graduate) (a) For $z = x + iy$, we have by Theorem 1.5.1 that $x = \dfrac{z+\overline{z}}{2}$ and $y = \dfrac{z-\overline{z}}{2i}$. So with $F(x, y)$ as a complex valued function of two real variables, *formally* apply the Chain Rule to $F(x, y)$ (see Theorem 5 in 14.4. The Chain Rule from my online Calculus 3 notes) to show

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

Note. We say "*formally* apply" because the Chain Rule mentioned above applies to real valued functions of two real variables.

(b) Define the operator $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$, "suggested" by part (a), to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}((u_x - v_y) + i(v_x + u_y)) = 0.$$

This is the *complex form* $\partial f/\partial \overline{z} = 0$ of the Cauchy-Riemann equations. That is, the equation $\partial f/\partial \overline{z} = 0$ is equivalent to the conditions that $u_x = v_y$ and $u_y = -v_x$.