Chapter 1. Complex Numbers

Note. This chapter explores some algebraic and geometric properties of complex numbers. Functions and their derivatives are introduced in Chapter 2, more functions and their inverses are given in Chapter 3, integrals are studies in Chapter 4, and Series are addressed in Chapter 5. This complex variables class will cover these five chapters and possibly additional material from later in the book.

Section 1. Sums and Products

Note. There are a number of ways to define the field of complex numbers, \( \mathbb{C} \). One way is to define it as an extension field of the real numbers, \( \mathbb{C} = \mathbb{R}[i] \) where \( i \) is a root of the real polynomial \( x^2 + 1 \) (see the last page of my notes from Introduction to Modern Algebra [MATH 4127/5127], Section 29 on Extension fields: http://faculty.etsu.edu/gardnerr/4127/notes/VI-29.pdf; see that last page).

Definition. The field of complex numbers, \( \mathbb{C} \), is the set of ordered pairs of real numbers: \( \mathbb{C} = \{(x, y) \mid x, y \in \mathbb{R}\} \), where addition is defined as

\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)
\]

and multiplication is defined as

\[
(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2).
\]
Note. As you might expect, we denote \( z = (x, y) \in \mathbb{C} \) as \( z = x + iy \). Geometrically, \( \mathbb{C} \) is the same as the Cartesian plane, \( \mathbb{R}^2 \) (however they are different algebraically; we do not multiply elements of \( \mathbb{R}^2 \), say... though if we take \( \mathbb{R}^2 \) as a vector space, then we can add elements). So we can associate any element of \( \mathbb{C} \) with a point in \( \mathbb{R}^2 \). When doing so, we call the \( x \)-axis the “real axis” and call the \( y \)-axis the “imaginary axis.” We have:

![Diagram of complex plane]

For \( z \in \mathbb{C} \), we denote \( x = \text{Re}(z) \) (the “real part of \( z \)” ) and \( y = \text{Im}(z) \) (the “imaginary part of \( z \)” ). An older notation is \( x = \Re(z) \) and \( y = \Im(z) \). Notice that \( z = \text{Re}(z) + i\text{Im}(z) \), where \( i = (0, 1) \).

Note. With \( (0, 1) \) denoted as \( i \), we have by the definition of multiplication that

\[
i^2 = (0, 1)(0, 1) = ((0)(0) - (1)(1), (1)(0) + (0)(1)) = (-1, 0) = -1 + i0 = -1.
\]
Note. By writing $(x, y) = x + iy$, the definition of addition in $\mathbb{C}$ becomes

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

and the definition of multiplication becomes

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

$$= (x_1)(x_2) + (x_1)(iy_2) + (iy_1)(x_2) + (iy_1)(iy_2) \text{ ("FOIL")}.$$ 

So addition and multiplication in $\mathbb{C}$ the usual (field) properties. We now abandon the ordered pair notation (though we will return to the geometric presentation of $\mathbb{C}$ in terms of $\mathbb{R}^2$) and use the notation $z = x + iy$ introduced above. When we do appeal to the geometric plane representation, we call this the “complex plane” (and use this term synonymously with the terms “complex field” or “complex numbers”).

Revised: 8/22/2016