Section 1.11. Regions in the Complex Plane

Note. In order to deal with the calculus of functions of a complex variable, we need to take the $\varepsilon/\delta$ ideas from Calculus 1 and 2 and apply them to the complex setting. This is largely accomplished by replacing the distance measure on $\mathbb{R}$ to the distance measure on $\mathbb{C}$. Recall that the distance between $x_1, x_2 \in \mathbb{R}$ is $|x_1 - x_2|$ (the absolute value of the difference). We have seen that the distance between $z_1, z_2 \in \mathbb{C}$ is $|z_1 - z_2|$ (the modulus of the difference).

Note. To inspire the things we define in this section, let’s recall the definition of “limit” from Calculus 1: “Let $f(x)$ be defined on an open interval about $x_0$, except possibly at $x_0$ itself. We say that $f(x)$ approaches the limit $L$ as $x$ approaches $x_0$ and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $x$,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$"

Definition. For a given $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, the set $\{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$ is called an $\varepsilon$ neighborhood of $z_0$, which we denote simply as “$|z - z_0| < \varepsilon$.” The deleted $\varepsilon$ neighborhood of $z_0$ is the set $\{z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon\}$, abbreviated “$0 < |z - z_0| < \varepsilon$.”
Definition. A point $z_0$ is an interior point of set $S \subset \mathbb{C}$ if there is some $\varepsilon$ neighborhood of $z_0$ which is a subset of $S$. A point $z_0$ is an exterior point of a set $S \subset \mathbb{C}$ if there is some $\varepsilon$ neighborhood of $z_0$ containing no points of $S$ (i.e., disjoint from $S$). A point $z_0$ is a boundary point of set $S \subset \mathbb{C}$ if it is neither an interior point nor an exterior point of $S$. The set of all boundary points of set $S$ is called the boundary of $S$, sometimes denoted $\partial(S)$.

Lemma 1.11.A. A point $z_0$ is a boundary point of set $S$ if and only if every $\varepsilon$ neighborhood of $z_0$ contains at least one point in set $S$ and at least one point not in $S$.

Definition. A set of complex numbers is open if it contains none of its boundary points. A set of complex numbers is closed if it contains all of its boundary points. The closure of set $S \subset \mathbb{C}$ is the set consisting of all points of $S$ and all boundary points of $S$.

Definition. An open set $S \subset \mathbb{C}$ is connected if each pair of points $z_1, z_2 \in S$ can be joined by a polygonal line consisting of a finite number of line segments joined end to end, that lies entirely in $S$. A nonempty open connected set is a domain. A domain together with some, none, or all of its boundary points is a region.
Note. The annulus $1 < |z| < 2$ is an open connected set, as suggested in Figure 16.

Definition. A set $S \subset \mathbb{C}$ is \textit{bounded} if $S$ lies in some circle $|z| = R$. A set of complex numbers that is not bounded is \textit{unbounded}.

Definition. A point $z_0$ is an \textit{accumulation point} of set $S \subset \mathbb{C}$ if each deleted neighborhood of $z_0$ contains at least one point of $S$.

Lemma 1.11.B. If a set $S \subset \mathbb{C}$ is closed, then $S$ contains all of its accumulation points.

Definition. A point $z_0 \in S$ is an \textit{isolated point} of set $S$ if there is a deleted neighborhood of $z_0$ containing no points in set $S$ (i.e., disjoint from $S$).
Example 1.11.A. Find the set of interior points, boundary points, accumulation points, and isolated points for:

The interior points are all points in the set \( \{ z \in \mathbb{C} \mid |z| < 1 \} \). The boundary points are all points in the set \( \{ z \in \mathbb{C} \mid |z| = 1 \} \cup \{2\} \). The accumulation points are all points in the set \( \{ z \in \mathbb{C} \mid |z| \leq 1 \} \). The only isolated point is \( z = 2 \).

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