Section 1.11. Regions in the Complex Plane

Note. In order to deal with the calculus of functions of a complex variable, we need to take the ε/δ ideas from Calculus 1 and 2 and apply them to the complex setting. This is largely accomplished by replacing the distance measure on $\mathbb{R}$ to the distance measure on $\mathbb{C}$. Recall that the distance between $x_1, x_2 \in \mathbb{R}$ is $|x_1 - x_2|$ (the absolute value of the difference). We have seen that the distance between $z_1, z_2 \in \mathbb{C}$ is $|z_1 - z_2|$ (the modulus of the difference).

Note. To inspire the things we define in this section, let's recall the definition of "limit" from Calculus 1 (see also my online Calculus 1 [MATH 1910] notes on Section 2.3. The Precise Definition of a Limit): "Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the *limit L as x approaches x_0* and write $\lim_{x \to x_0} f(x) = L$, if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \text{ implies } |f(x) - L| < \varepsilon.\text{"}$$

Note. We now give several definitions and theorems related to "topological properties" of sets of complex numbers. In the real setting, this is covered in Analysis 1 (MATH 4217/5217) in Section 3.1. Topology of the Real Numbers. A more general presentation of these ideas is given in Introduction to Topology (MATH 4357/5357).

Definition. For a given $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, the set $\{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$ is called an ε *neighborhood* of z_0, which we denote simply as "$|z - z_0| < \varepsilon$." The *deleted* ε *neighborhood* of z_0 is the set $\{z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon\}$, abbreviated "$0 < |z - z_0| < \varepsilon$."

Definition. A point z_0 is an *interior point* of set $S \subset \mathbb{C}$ if there is some ε neighborhood of z_0 which is a subset of S. A point z_0 is an *exterior point* of a set $S \subset \mathbb{C}$ if there is some ε neighborhood of z_0 containing no points of S (i.e., disjoint from S). A point z_0 is a *boundary point* of set $S \subset \mathbb{C}$ if it is neither an interior point nor an exterior point of S. The set of all boundary points of set S is called the *boundary* of S, sometimes denoted $\partial(S)$.

Lemma 1.11.A. A point z_0 is a boundary point of set S if and only if every ε neighborhood of z_0 contains at least one point in set S and at least one point not in S.

Definition. A set of complex numbers is *open* if it contains none of its boundary points. A set of complex numbers is *closed* if it contains all of its boundary points. The *closure* of set $S \subset \mathbb{C}$ is the set consisting of all points of S and all boundary points of S.

Note 1.11.A. For a given set S, there are three kinds of points: interior points, exterior points, and boundary points. So if S is open, then every point is an interior point. That is, S is open if and only if for any $z_0 \in S$ there is $\varepsilon > 0$ such that the ε neighborhood of z_0 is a subset of S. This is the approach taken when defining a set as "open" in a metric space. For example, this is the approach taken in Analysis 1 (MATH 4217/5217) for $\mathbb{R}$; see my online Analysis 1 notes on Section 3.1. Topology of the Real Numbers where an ε neighborhood of $x_0 \in \mathbb{R}$ is an open interval of the form $(x_0 - \varepsilon, x_0 + \varepsilon)$.

Definition. An open set $S \subset \mathbb{C}$ is *connected* if each pair of points $z_1, z_2 \in S$ can be joined by a *polygonal line* consisting of a finite number of line segments joined end to end, that lies entirely in S. A nonempty open connected set is a *domain*. A domain together with some, none, or all of its boundary points is a *region*.

Note. In graduate level Complex Analysis 1 (MATH 5510), a different approach is taken to connectedness. A "separation" of a set is defined and used to define a connected set in a metric space. The definition above then follows as a theorem. See my online notes for Complex Analysis 1 on Section II.2. Connectedness; notice Note II.2.A and Theorem II.2.3. A separation is also used in Analysis 1 (see my online notes for Analysis 1 on Section 3.1. Topology of the Real Numbers; notice Note 3.1.M) and in the general topological space setting (see my online notes for Introduction to Topology on Section 23. Connected Spaces).

Note. The annulus $1 < |z| < 2$ is an open connected set, as suggested in Figure 16.

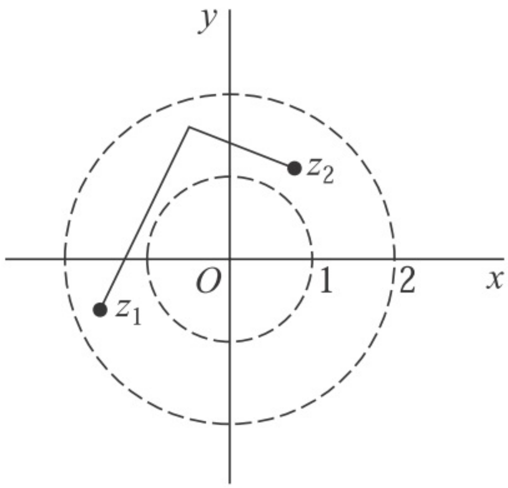


FIGURE 16

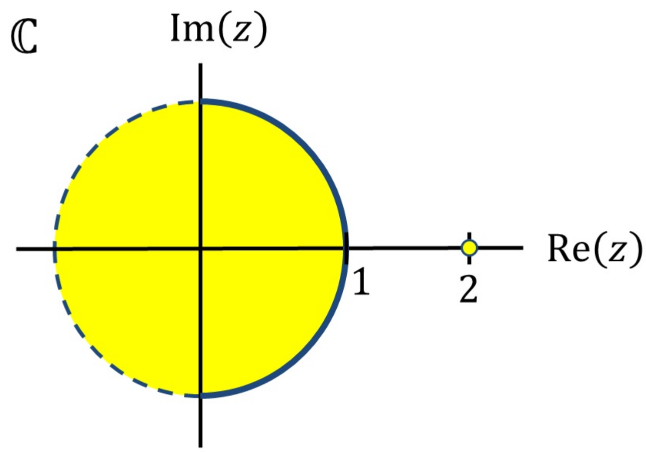
Definition. A set $S \subset \mathbb{C}$ is *bounded* if S lies in some circle $|z| = R$. A set of complex numbers that is not bounded is *unbounded*.

Definition. A point z_0 is an *accumulation point* of set $S \subset \mathbb{C}$ if each deleted neighborhood of z_0 contains at least one point of S.

Lemma 1.11.B. If a set $S \subset \mathbb{C}$ is closed, then S contains all of its accumulation points.

Definition. A point $z_0 \in S$ is an *isolated point* of set S if there is a deleted neighborhood of z_0 containing no points in set S (i.e., disjoint from S).

Example 1.11.A. Find the set of interior points, boundary points, accumulation points, and isolated points for:



The interior points are all points in the set $\{z \in \mathbb{C} \mid |z| < 1\}$. The boundary points are all points in the set $\{z \in \mathbb{C} \mid |z| = 1\} \cup \{2\}$. The accumulation points are all points in the set $\{z \in \mathbb{C} \mid |z| \leq 1\}$. The only isolated point is $z = 2$.

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