Section 1.3. Further Properties

Note. In this brief section, we introduce a few formulas (as lemmas) and the Binomial Theorem.

Lemma 1.3.1. For any \( z_1, z_2, z_3 \in \mathbb{C} \), with \( z_3 \neq 0 \), we have

\[
\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}.
\]

Proof. We have

\[
\frac{z_1 + z_2}{z_3} = (z_1 + z_2)z_3^{-1} \quad \text{by the definition of division in } \mathbb{C}
\]
\[
= z_1z_3^{-1} + z_2z_3^{-1} \quad \text{by Theorem 1.2.1(3) (distribution)}
\]
\[
= \frac{z_1}{z_3} + \frac{z_2}{z_3} \quad \text{by the definition of division in } \mathbb{C}.
\]

Note. Lemma 1.3.1 deals with addition of quotients and “common denominators.” The following relates inverses and products (“the products of inverses is the inverse of the products”).

Lemma 1.3.2. If \( z_1, z_2 \in \mathbb{C} \), with \( z_1 \neq 0, z_2 \neq 0 \), then

\[
\left( \frac{1}{z_1} \right) \left( \frac{1}{z_2} \right) = \frac{1}{z_1z_2},
\]

or equivalently, \( z_1^{-1}z_2^{-1} = (z_1z_2)^{-1} \).
Proof. We have

\[(z_1 z_2)(z_2^{-1} z_1^{-1}) = (z_1 z_2)(z_1^{-1} z_2^{-1})\] by Theorem 1.2.1(2), commutivity of multiplication

\[= (z_1 z_2) z_2^{-1} z_2\]

\[= z_1 (z_2 z_2^{-1}) z_1^{-1}\] by Theorem 1.2.1(2), associativity of multiplication

\[= z_1 (1) z_1^{-1}\] by the definition of multiplicative inverse

\[= z_1 z_1^{-1}\] since 1 is the multiplicative identity

\[= 1\] by the definition of multiplicative inverse.

Since \((z_1 z_2)(z_1^{-1} z_2^{-1}) = 1\) then \((z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}\).

Theorem 1.3.2. The Binomial Theorem.

For any \(z_1, z_2, z_3 \in \mathbb{C}\) and \(n \in \mathbb{N} = \{1, 2, 3, \ldots\}\) (the natural numbers) we have

\[(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^n z_2^{n-k}\]

where

\[\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2) \cdots (n-k+1)!}{k(k-1)(k-2) \cdots (3)(2)(1)}\]

and \(0! = 1\) (by definition).

Note. You are asked to prove the Binomial Theorem in Exercise 1.3.8 using Mathematical Induction. This is similar to the proof you likely saw in Mathematical Reasoning (MATH 3000).

Revised: 1/23/2020