Section 1.9. Roots of Complex Numbers

Note. We now use the results of the previous two sections to find $n$th roots of complex numbers. In $\mathbb{R}$, there are two “choices” for a square root of $x$ when $x > 0$ (a positive square root and a negative square root). This problem is compounded in the complex setting by the fact that there are $n$ “choices” for the $n$th root of a nonzero complex number.

Note. You may have seen “$n$th roots of unity” in Introduction to Modern Algebra (MATH 4127/5127; see my online class notes on I. Groups and Subgroups, Section 1. Introduction and Examples.). The $n$th roots of unity form a cyclic group of order $n$ under multiplication.

Note. Since the function $e^{i\theta}$ is a periodic function in $\theta$ with period $2\pi$ (in fact, as a function of $\theta$ graphed in the complex plane, this function traces out the unit circle $|z| = 1$). So if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ (where $r_1 > 0$ and $r_2 > 0$) then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$ for some $k \in \mathbb{Z}$.

Note. Suppose $z_0 = r_0 e^{i\theta_0}$ and $z^n = z_0$ where $z = re^{i\theta}$. Then it must be that $z^n = r^n e^{in\theta} = z_0 = r_0 e^{i\theta_0}$ and so $r^n = r_0$ and $n\theta = \theta_0 + 2k\pi$ for some $k \in \mathbb{Z}$. So we must have $r = \sqrt[n]{r_0}$ and $\theta = (\theta_0 + 2k\pi)/n$ for $k \in \mathbb{Z}$. Therefore, the $n$th roots of $z_0$ are $z = \sqrt[n]{r_0} \exp(i(\theta_0 + 2k\pi)/n)$ for $k \in \mathbb{Z}$. However, since $\exp(i\theta)$ is periodic,
there are in fact only \( n \) distinct \( n \)th roots of \( z_0 \). Namely

\[
c_k = \sqrt[n]{r_0} \exp \left( i \frac{\theta_0 + 2k\pi}{n} \right) \text{ for } k = 0, 1, \ldots, n - 1.
\]

With \( n \geq 3 \), the roots lie at the vertices of a regular \( n \)-gon inscribed in a circle of radius \( \sqrt[n]{r} \) and centered at 0 (we’ll have illustrations of this in the next section). When \( \theta_0 \) is the principal argument of \( z \) then \( c_0 \) is the *principal \( n \)th root* of \( z \).

**Note.** If \( z = 1 = 1e^{i0} \), we get the “\( n \)th roots of unity”

\[
\omega_n^k = \exp \left( i \frac{2k\pi}{n} \right) \text{ for } k = 0, 1, \ldots, n - 1.
\]

Notice that \( \omega_n^1 = \exp(i2\pi/n) \) can be used to generate each of the other \( n \)th roots of unity: \( \omega_n^k = (\omega_n^1)^k \). This is how we can form a cyclic group out of the \( n \)th roots of unity and \( \omega_n^1 \) is a generator of this cyclic group (which is isomorphic to \( \langle \mathbb{Z}_n, + \rangle \)).

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