Section 2.14. Mappings by the Exponential Function

Note. We take as the definition of $e^z$ the following: $e^z = e^{x+iy} = e^x e^{iy}$. Of course $e^x$ for $x \in \mathbb{R}$ is defined and $e^{iy}$ is defined by Euler’s formula $e^{iy} = \cos y + i \sin y$. In graduate-level Complex Analysis 1 (MATH 5510), the properties of power series are studied and $e^z$ is defined in terms of a series (see my online notes on III.1. Power Series). The purpose of this section is to explore some mapping properties implied by the above definition.

Example 2.14.1. In polar coordinates $w = \rho e^{i\varphi}$ we have from $e^z = e^{x+iy} = e^x e^{iy}$ that $\rho = e^x$ and $\varphi = y$. So a point $z = c_1 + iy$ on the vertical line $x = c_1$ in the $z$-plane is mapped by $f(z) = e^z$ to the point $w = \rho e^{i\varphi} = e^{c_1} e^{iy}$ in the $w$-plane which lies on the circle $\rho = e^{c_1}$ in the $w$-plane and as $y$ varies (as the point $z = c_1 + iy$ moves along the vertical line $x = c_1$ in the $z$-plane) the image $w$ moves around the circle $\rho = e^{c_1}$ as shown in Figure 20. Notice that geometrically the line is mapped around the circle an infinite number of times. The horizontal line $y = c_2$ in the $z$-plane is mapped by $f(z) = e^z$ onto the ray $\varphi = c_2$ and $\rho > 0$. See Figure 20. This is a one to one mapping.
Example 2.14.2. Consider the rectangle in the \( z \)-plane \( \{x + iy \in \mathbb{C} \mid a \leq x \leq b, c \leq y \leq d\} \). Then \( f(z) = e^z \) maps this to \( \{\rho e^{i\varphi} \in \mathbb{C} \mid e^a \leq \rho \leq e^b, c \leq \varphi \leq d\} \). See Figure 21.

![Figure 21](image)

**FIGURE 21** \( w = \exp z \).

Example 2.14.3. Since horizontal lines are mapped to rays, the strip \( 0 \leq y \leq \pi \) in the \( z \)-plane is mapped to the upper half-plane \( v \geq 0 \) of the \( w \)-plane (except for \( 0 \in w \)). Also, any horizontal strip of height \( 2\pi \) (and infinite width) is mapped to the entire \( w \)-plane except 0. Therefore \( f(z) = e^z \) maps \( \mathbb{C} \) to \( \mathbb{C} \) in an infinite to 1 way (in fact, \( f(z) = e^z \) is a periodic function of period \( 2\pi i \)). This has intense implications in defining an inverse of \( e^z \).
Note. In Figures 6, 7, and 8 of Appendix 2, these mappings are further illustrated.