Section 2.17. Limits Involving the Point at Infinity

Note. In this section, we introduce the symbol $\infty$ and rigorously define limits of $f(z)$ as $z$ approaches $\infty$ and limits of $f(z)$ which are $\infty$.

Note. Brown and Churchill introduce a sphere of radius 1 centered at the origin of the complex plane. They define the point $N$ as the point on the sphere farthest from the plane and “above” the plane (corresponding, in a sense, to the “north pole” of the sphere). They then map each point $P$ on the sphere (other than $N$) onto the plane by projecting the point $P$ onto the plane with a straight line through $P$ and $N$. This is called the stereographic projection and the sphere is called the Riemann sphere. The point $N$ itself is then associated with the symbol $\infty$. In this way, we have a one to one and onto mapping (i.e., a bijection) from the Riemann sphere to $\mathbb{C} \cup \{\infty\}$ (which is called the extended complex plane).
Note. In Introduction to Topology (MATH 4357/5357), you will encounter the extended complex plane as a “one-point compactification” of the complex plane; see my online notes for Introduction to Topology at 29. Local Compactness (see Example 4). We also address the extended complex plane as a metric space in our graduate-level Complex Analysis 1 class (MATH 5510); see my notes for this class at I.6. The Extended Plane and its Spherical Representation.

Definition. In the extended complex plane, an \( \varepsilon \)-neighborhood of \( \infty \) is the set \( \{ z \in \mathbb{C} | \frac{1}{|z|} < \varepsilon \} \). An open set containing an \( \varepsilon \)-neighborhood of \( \infty \) for some \( \varepsilon > 0 \) is a neighborhood of \( \infty \).

Definition. Let \( f \) be a function defined and nonzero at all points \( z \) of some neighborhood of \( \infty \). If there is \( w_0 \in \mathbb{C} \) such that for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
1/|z| < \delta \quad \text{implies} \quad |f(z) - w_0| < \varepsilon,
\]

then the limit as \( z \) approaches \( \infty \) of \( f \) is \( w_0 \), denoted \( \lim_{z \to \infty} f(z) = w_0 \).

Definition. Let \( f \) by a function defined and nonzero at all points \( z \) in some deleted neighborhood of \( z_0 \). If for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
0 < |z - z_0| < \delta \quad \text{implies} \quad 1/|f(z)| < \varepsilon,
\]

then the limit of \( f \) as \( z \) approaches \( z_0 \) is \( \infty \), denoted \( \lim_{z \to z_0} f(z) = \infty \).
**Definition.** Let $f$ be a function defined and nonzero at all points $z$ of some neighborhood of $\infty$. If for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$1/|z| < \delta \text{ implies } 1/|f(z)| < \varepsilon$$

then the *limit as $z$ approaches $\infty$* of $f$ is $\infty$, denoted $\lim_{z \to \infty} f(z) = \infty$.

**Theorem 2.17.1.** If $z_0, w_0 \in \mathbb{C}$ then

\[
\begin{align*}
\lim_{z \to z_0} f(z) &= \infty \text{ if and only if } \lim_{z \to z_0} 1/f(z) = 0 \\
\lim_{z \to \infty} f(z) &= w_0 \text{ if and only if } \lim_{z \to 0} f(1/z) = w_0, \text{ and} \\
\lim_{z \to \infty} f(z) &= \infty \text{ if and only if } \lim_{z \to 0} 1/f(1/z) = 0.
\end{align*}
\]

**Example 2.17.A.** We now establish the following limits.

(a) $\lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty$.

**Solution.** We let $f(z) = \frac{iz + 3}{z + 1}$ and consider

\[
\lim_{z \to -1} \frac{1}{f(z)} = \lim_{z \to -1} \frac{z + 1}{iz + 3} = \frac{(-1) + 1}{i(-1) + 3} = \frac{0}{3 - i} = 0
\]

where we have evaluated the limit using Corollary 2.16.B. So by Theorem 2.17.1 (1st claim), $\lim_{z \to -1} f(z) = \lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty$. □

(b) $\lim_{z \to \infty} \frac{2z + i}{z + 1} = 2$.

**Solution.** We let $f(z) = \frac{2z + i}{z + 1}$ and consider

\[
\lim_{z \to \infty} f(1/z) = \lim_{z \to 0} \frac{2(1/z) + 1}{(1/z) + 1} = \lim_{z \to 0} \frac{2(1/z) + 1}{1 + z} = \lim_{z \to 0} \frac{2 + iz}{1 + z} = \frac{2 + i(0)}{1 + (0)} = \frac{2}{1} = 2
\]
where we have evaluated the limit using Corollary 2.16.B. So by Theorem 2.17.1 (2nd claim), \( \lim_{z \to \infty} f(z) = \lim_{z \to \infty} \frac{2z + i}{z + 1} = 2. \square \)

(c) \( \lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1} = \infty. \)

**Solution.** We let \( f(z) = \frac{2z^3 - 1}{z^2 + 1} \) and consider

\[
\lim_{z \to 0} \frac{1}{f(1/z)} = \lim_{z \to 0} \frac{(1/z)^2 + 1}{2(1/z)^3 - 1} = \lim_{z \to 0} \frac{(1/z)^2 + 1}{2(1/z)^3 - 1} \cdot \frac{z^3}{z^3} = \lim_{z \to 0} \frac{z + z^3}{2 - z^3}
\]

\[
= \lim_{z \to 0} \frac{(0) + (0)^3}{2 - (0)^3} = 0
\]

where we have evaluated the limit using Corollary 2.16.B. So by Theorem 2.17.1 (3rd claim), \( \lim_{z \to \infty} f(z) = \infty. \square \)