Section 2.19. Derivatives

Note. Our definition of derivative is the same as in Calculus 1 (although our definition of limit is arguably more restrictive).

Definition. Let \( f \) have a domain which contains a neighborhood of \( z_0, \{ z \mid |z - z_0| < \varepsilon \} \) for some \( \varepsilon > 0 \). The derivative of \( f \) at \( z_0 \) is

\[
f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},
\]

provided the limit exists. Function \( f \) is differentiable at \( z_0 \) when \( f'(z_0) \) exists. We can define the derivate of \( f \) as a function of the \( z \)-values where the derivative is defined and denote this function as \( f'(z) \), the derivative of \( f \). We often let \( w = f(z) \) and denote \( f'(z) = \frac{dw}{dz} \).

Example 2.19.1. Let \( f(z) = z^2 \). Then

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z)}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z + 0 = 2z.
\]
Example 2.19.2. Let \( f(z) = \overline{z} \), then
\[
\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{z + \Delta z - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta \overline{z}}{\Delta z}.
\]

As in Note 2.15.A, we can apply the Two-Path Test for Nonexistence of a Limit and consider the limit \( \Delta z = \Delta x + i\Delta y \to 0 \) along the real axis, \( \Delta z = \Delta x + i(0) = \Delta x \to 0 \), and along the imaginary axis, \( \Delta z = 0 + i\Delta y = i\Delta y \to 0 \), as in Figure 29.

Along these paths we have
\[
\lim_{\Delta x \to 0} \frac{\Delta \overline{z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1
\]
and
\[
\lim_{i\Delta y \to 0} \frac{\Delta \overline{z}}{\Delta z} = \lim_{i\Delta y \to 0} \frac{i\Delta y}{i\Delta y} = \lim_{i\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = \lim_{\Delta x \to 0} -1 = -1.
\]
Since the two paths give different limits, then the limit cannot exist. That is, \( f(z) = \overline{z} \) is not differentiable at any \( z \in \mathbb{C} \). This approach to studying differentiability foreshadows our approach in Section 2.21 ("The Cauchy-Riemann Equations").
Example 2.19.3. Let \( f(z) = |z|^2 \), then
\[
\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z\overline{z} + \Delta z\overline{z} + z\overline{\Delta z} + \Delta z\overline{\Delta z} - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \left( \frac{z + \overline{\Delta z} + z\overline{z}}{\Delta z} \right).
\]
As in Example 2.19.2, using the Two-Path Test,
\[
\lim_{\Delta x \to 0} \left( \frac{|\overline{\Delta z} + z\overline{\Delta z}|}{\Delta z} \right) = \lim_{\Delta x \to 0} \left( \frac{|\overline{\Delta x} + z\overline{\Delta x}|}{\Delta x} \right) = \lim_{\Delta x \to 0} \left( \frac{z + \overline{\Delta x} + z\overline{\Delta x}}{\Delta x} \right)
\]
and
\[
\lim_{i\Delta y \to 0} \left( \frac{|\overline{\Delta z} + z\overline{\Delta z}|}{\Delta z} \right) = \lim_{i\Delta y \to 0} \left( \frac{|\overline{\Delta y} + z\overline{\Delta y}|}{i\Delta y} \right) = \lim_{i\Delta y \to 0} \left( \frac{z - i\Delta y + z(-1)}{i\Delta y} \right) = \lim_{i\Delta y \to 0} (\overline{z} - i\Delta y + z(-1)) = \overline{z} - z,
\]
Now these two limits, \( \overline{z} + z \) and \( \overline{z} - z \), are not equal unless \( z = 0 \). So if \( z \neq 0 \) then the limits are different and \( f(z) = |z|^2 \) is not differentiable. Next, we must check \( z = 0 \):
\[
\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 0.
\]
So \( f'(0) = 0 \) and \( f \) is differentiable at \( z = 0 \).

Note. Example 2.19.3 shows that a function \( f \) can be continuous at every point \( (f(z) = |z|^2 \) or \( f(x + iy) = x^2 + y^2 \) is continuous at all \( z = x + iy \)) but may not be differentiable. However, we have, as in Calculus 1, the following.
Theorem 2.19.A. Differentiable implies Continuous. If $f$ is differentiable at point $z_0$ then $f$ is continuous at $z_0$. 