Section 2.21. Cauchy-Riemann Equations

Note. In this section, we introduce the Cauchy-Riemann equations, which are a necessary condition for the differentiability of a function \( f \) at a point \( z_0 \). In the next section, we show that these equations, along with some added continuity hypotheses, are sufficient for the differentiability of function \( f \) at point \( z_0 \).

Theorem 2.21.A. Differentiable Implies the Cauchy-Riemann Equations

Suppose that \( f(z) = u(x, y) + iv(x, y) \) and that \( f' \) exists at a point \( z_0 = x_0 + iy_0 \). Then the first-order partial derivatives of \( u \) and \( v \) must exist at \( (x_0, y_0) \), and they must satisfy the Cauchy-Riemann equations:

\[
\frac{\partial}{\partial x}[u(x, y)] = \frac{\partial}{\partial y}[v(x, y)] \quad \text{and} \quad \frac{\partial}{\partial y}[u(x, y)] = -\frac{\partial}{\partial x}[v(x, y)]
\]

(or with subscripts representing partial derivatives, \( u_x = v_y \) and \( u_y = -v_x \)) at \( (x_0, y_0) \). Also, \( f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \).

Note. The proof is rather elementary and results from considering a limit as \( \Delta x \to 0 \) along the real axis and along the imaginary axis. We now present the proof.

Example 2.21.1. In Example 2.19.1, we saw that for \( f(z) = z^2 \) that \( f'(z) = 2z \) and that this holds for all \( z \in \mathbb{C} \). So \( f \) must satisfy the Cauchy-Riemann equations at all points \( (x, y) \). First, \( f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy) \) so that \( u(x, y) = x^2 - y^2 \) and \( v(x, y) = 2xy \). We then have \( u_x = 2x = v_y \) and \( u_y = -2y = -v_x \), so the Cauchy-Riemann equations are satisfied. We also have by Theorem 2.21.A that \( f'(z) = u_x + iv_x = 2 + i(2y) = 2z \).

Example 2.21.2. Since the Cauchy-Riemann equations are necessary for (pointwise) differentiability, we can use them to find points where \( f \) is not differentiable. Consider \( f(z) = |z|^2 = |x + iy|^2 = (x^2 + y^2) + i(0) \). We have \( u(x, y) = x^2 + y^2 \) and \( v(x, y) = 0 \). So \( u_x = 2x, u_y = 2y, \) and \( v_x = v_y = 0 \). So for the Cauchy-Riemann equations to be satisfied, we need \( x = 0 = y \). That is, the only point at which \( f \) may be differentiable is at \( z_0 = 0 + 0i \). Once we see the sufficient conditions for differentiability, then we will see that \( f(z) = |z|^2 \) is in fact differentiable at \( z_0 = 0 \) and that \( f'(0) = 0 \).

Revised: 1/3/2020