Section 2.22. Sufficient Conditions for Differentiability

Note. In the previous section, we saw that if function \( f \) is differentiable point \( z_0 \) then the Cauchy-Riemann equations must be satisfied at \( z_0 \). In this section, we show that the Cauchy-Riemann equations are sufficient for the differentiability of function \( f \) at point \( z_0 \) with some added continuity conditions on the partial derivatives of \( u(x, y) \) and \( v(x, y) \).

Note. Consider the function

\[
f(z) = \begin{cases} 
\bar{z}^2/z & \text{when } z \neq 0 \\
0 & \text{when } z = 0.
\end{cases}
\]

In Exercise 2.20.9 (also Exercise 2.20.9 in the 9th edition of the book) it is shown that \( f'(0) \) does not exist. However, in Exercise 2.23.6 (Example 2.22.3 in the 9th edition of the book) it is shown that \( f \) satisfies the Cauchy-Riemann equations at \( z_0 = 0 \). So the Cauchy-Riemann equations are not, alone, sufficient to guarantee the differentiability of \( f \). We now give sufficient conditions for the differentiability of \( f \) at \( z_0 \).

**Theorem 2.22.A. The Cauchy-Riemann Equations and Continuity Imply Differentiability**

Let the function \( f(z) = u(x, y) + iv(x, y) \) be defined throughout some \( \varepsilon \) neighborhood of a point \( z_0 = x_0 + iy_0 \), and suppose that

(a) the first-order partial derivatives of the functions \( u \) and \( v \) with respect to \( x \) and \( y \) exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at \( (x_0, y_0) \) and satisfy the Cauchy-Riemann equations \( u_x(x_0, y_0) = v_y(x_0, y_0) \) and \( y_y(x_0, y_0) = -v_x(x_0, y_0) \).
Section 2.22. Sufficient Conditions for Differentiability

Then \( f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \).

**Example 2.22.1.** Consider \( f(x) = e^z = e^x e^{iy} \) (where \( z = x + iy \)). By Euler’s formula, we have \( f(z) = e^x \cos y + ie^x \sin y \), so \( u(x, y) = e^x \cos y \) and \( v(x, y) = e^x \sin y \). Since \( u_x = e^x \cos y = v_y \) and \( u_y = -e^x \sin y = -v_x \) and these derivatives are continuous for all \((x, y)\) then, by Theorem 2.22.A, \( f'(z) = u_x(u, y) + iv_x(x, y) = e^x \cos y + ie^x \sin y = e^z \) for all \( z \in \mathbb{C} \). This is consistent with your ideas from Calculus 2 and will be verified again when we consider power series in Chapter 5.

**Example 2.22.2.** We saw in Example 2.21.2 that \( f(z) = |z|^2 \) satisfies the Cauchy Riemann equations at \( z_0 = 0 \). Since \( f(z) = |z|^2 = (x^2 + y^2) + i(0) \), then \( u(x, y) = x^2 + y^2 \) and \( v(x, y) = 0 \) then the first-order partial derivatives of \( u \) and \( v \) exist and are continuous everywhere, so by Theorem 2.22.A, \( f \) is differentiable at \( z_0 = 0 \).

*Revised: 1/3/2019*