Section 2.23. Polar Coordinates

**Note.** In this section, we restate the results of the previous two sections on the Cauchy-Riemann equations, but this time in polar coordinates \((r, \theta)\) instead of rectangular coordinates \((x, y)\).

**Lemma 2.23.A.** Let the function \(f(z) = u(x, y) + iv(x, y)\) be defined throughout some \(\epsilon\) neighborhood of a point \(z_0 = x_0 + iy_0\), and suppose that

(a) the first-order partial derivatives of the functions \(u\) and \(v\) with respect to \(x\) and \(y\) exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at \((x_0, y_0)\) and satisfy the Cauchy-Riemann equations \(u_x(x_0, y_0) = v_y(x_0, y_0)\) and \(u_y(x_0, y_0) = -v_x(x_0, y_0)\).

Then with \(z_0 = r_0 \exp(i\theta_0) \neq 0\) we have

\[
 r_0 u_r(r_0, \theta_0) = v_{\theta}(r_0, \theta_0) \text{ and } u_{\theta}(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0).
\]

These are the polar coordinate forms of the Cauchy-Riemann equations.

**Lemma 2.23.B.** Let \(f(z) = f(r \exp(i\theta)) = u(r, \theta) + iv(r, \theta)\) be defined throughout some \(\epsilon\) neighborhood of a nonzero point \(z_0 = r_0 \exp(i\theta_0)\) and suppose that

(a) the first-order partial derivatives of the functions \(u\) and \(v\) with respect to \(r\) and \(\theta\) exist everywhere in the neighborhood;

(b) those partial derivatives are continuous at \((r_0, \theta_0)\) and satisfy the polar form \(ru_r = v_\theta\) and \(u_\theta = -rv_r\) of the Cauchy Riemann equations at \((r_0, \theta_0)\).
Then the Cauchy-Riemann equations in rectangular form are satisfied at $z_0 = x_0 + iy_0$:

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).$$

**Note.** The proof of Lemma 2.23.B is to be given in Exercise 2.23.7 (Exercise 2.24.5 in the 9th edition of the book).

**Lemma 2.23.C.** Let $f(z) = f(r \exp(i\theta)) = u(r, \theta) + iv(r, \theta)$ satisfy the hypotheses of Lemma 2.23.B. Then $f$ is differentiable at $z_0 = r_0 \exp(i\theta_0)$ and

$$f'(z_0) = e^{-i\theta_0}(u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)).$$

**Note.** The proof of Lemma 2.23.C is to be given in Exercise 2.23.8 (Exercise 2.24.6 in the 9th edition of the book). An alternative formula for $f'(z_0)$ is to be given in Exercise 2.23.9 (Exercise 2.24.7(a) in the 9th edition of the book):

$$f'(z_0) = \frac{-i}{z_0}(u_\theta(r_0, \theta_0) + iv_\theta(r_0, \theta_0)).$$

**Note.** Lemmas 2.23.A, 2.23.B, and 2.23.C combine to give the following.
Theorem 2.23.A. Let the function \( f(z) = f(r \exp(i\theta)) = u(r, \theta) + iv(r, \theta) \) be defined throughout some \( \varepsilon \) neighborhood of a point \( z_0 = r_0 \exp(i\theta_0) \), and suppose that

(a) the first-order partial derivatives of the functions \( u \) and \( v \) with respect to \( r \) and \( \theta \) exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at \( (r_0, \theta_0) \) and satisfy the polar form \( ru_r = v_\theta \) and \( u_\theta = -rv_r \) of the Cauchy-Riemann equations at \( (r_0, \theta_0) \).

Then \( f'(z_0) = e^{-i\theta_0}(u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)) \).

Example 2.23.2. Define \( f(z) = f(r \exp(i\theta)) = \sqrt[3]{r} \exp(i\theta/3) \) for \( r > 0, \alpha < \theta < \alpha + 2\pi \) for some fixed real \( \alpha \). This is “a cube root function.” We have

\[
u(r, \theta) = \sqrt[3]{r} \cos \left(\frac{\theta}{3}\right) \quad \text{and} \quad v(r, \theta) = \sqrt[3]{r} \sin \left(\frac{\theta}{3}\right), \quad \text{so}
\]

\[
u_r(r, \theta) = \frac{1}{3} r^{-2/3} \cos \left(\frac{\theta}{3}\right) = \frac{r^{1/3}}{3} \cos \left(\frac{\theta}{3}\right) = v_\theta(r, \theta) \quad \text{and}
\]

\[
u_\theta(r, \theta) = -\sqrt[3]{r} \sin \left(\frac{\theta}{3}\right) = -r \left(\frac{1}{3} r^{-2/3} \sin \left(\frac{\theta}{3}\right)\right) = -rv_r.
\]

So by Theorem 2.23.A, \( f \) is differentiable at all points at which it is defined and

\[
f'(z) = e^{-i\theta}(u_r(r, \theta) + iv_r(r, \theta)) = e^{-i\theta} \left(\frac{1}{3(\sqrt[3]{r})^2} \cos \left(\frac{\theta}{3}\right) + i \frac{1}{3(\sqrt[3]{r})^2} \sin \left(\frac{\theta}{3}\right)\right)
\]

\[
= \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2} e^{i\theta/3} = \frac{1}{3(\sqrt[3]{r} e^{i\theta/3})^2}.
\]

Notice that this derivative is similar to what we would expect if we were to differentiate the real cube root function: \( f(x) = x^{1/3} \) implies \( f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}} \).

We cannot explore root functions in detail until after we introduce the exponential function \( e^z \) and complex logarithm functions.

Revised: 3/14/2020