Section 2.25. Examples

Note. We now give some of the examples in the text, but present two of them as theorems.

Example 2.25.2. Consider \( f(z) = \cosh x \cos y + i \sinh x \sin y \). With \( f(z) = f(x+iy) = u(x, y) + iv(x, y) \) we have \( u(x, y) = \cosh x \cos y \) and \( v(x, y) = \sinh x \sin y \). Recall that \( \frac{d}{dx}[\cosh x] = \sinh x \) and \( \frac{d}{dx}[\sinh x] = \cosh x \) (see my Calculus 2 [MATH 1920] notes for more properties of the hyperbolic trig functions at 7.8. Hyperbolic Functions). We have that the Cauchy-Riemann equations are satisfied,

\[
u_x(x, y) = \sinh x \cos y = v_y \quad \text{and} \quad u_y(x, y) = -\cosh x \sin y = -v_x\]

for all \((x, y)\). Since the first partial derivatives of \( u \) and \( v \) exist and are continuous for all \((x, y)\), then by Theorem 2.22.A we have that \( f \) is an entire function.

Example 2.25.3. (Theorem 2.25.A.)

Suppose that a function \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) and its conjugate \( \overline{f(z)} = u(x, y) - iv(x, y) \) are both analytic in a given domain \( D \). Then \( f \) is constant throughout \( D \).

Example 2.25.4. (Theorem 2.25.B.)

Suppose that a function \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is analytic in a given domain \( D \) and that \(|f(z)|\) is constant throughout \( D \). Then \( f \) is constant throughout \( D \).
Note. We present one more theorem, but leave the proof as Exercise 2.25.7 (Exercise 2.26.7 in the 9th edition of the book).

Exercise 2.25.7. (Theorem 2.25.C.)
Let a function \( f \) be analytic everywhere in a domain \( D \). If \( f(z) \) is real-valued for all \( z \in D \), then \( f(z) \) must be constant throughout \( D \).

Revised: 1/4/2019