Section 3.30. The Logarithm Function

Note. In this section we begin defining the complex logarithm function; that is, defining the inverse of \( e^z \). This will provide some challenges since \( e^z \) is a periodic function with period \( 2\pi i \). The challenges are resolved with the definition of the principal branch of the logarithm.

Note. At the end of Section 3.29 we saw that for \( z \neq 0 \), \( z = e^\ln(|z| + i \arg(z)) \). The set \( \arg(z) \) consists of an infinite number of values, any two of which differ by an integer multiple of \( 2\pi i \). This leads to the following definition.

Definition. For any \( z \in \mathbb{C} \) with \( z \neq 0 \), define the “multiple-valued” function

\[
\log z = \ln |z| + i \arg(z).
\]

Note 3.30.A. Arguably, there is no such thing as a “multiple-valued” function by the very definition of “function.” Like the set valued “\( \arg(z) \)” encountered in Section 1.8, we can similarly think of \( \log z \) as defined here as set valued. Brown and Churchill approach it slightly differently. They take the unique value in \( \arg(z) \) in the interval \( (-\pi, \pi] \) and denote it as \( \Theta \). They then define

\[
\log z = \ln |z| + i(\Theta + 2n\pi) \text{ where } n \in \mathbb{Z}.
\]

This is equivalent to the definition above. Sometimes the value \( \Theta \) is called the \textit{principal argument} of \( z \). We’ll use it to define the principal branch of the logarithm. The interval \( (-\pi, \pi] \) is not universally accepted for the principal value. Notice that we have \( e^{\log z} = z \) for all \( z \neq 0 \).
Example. We saw in Section 3.29 that for all $z = x + iy = \ln(\sqrt{2}) + (\pi/4 + 2n\pi)i$ for $n \in \mathbb{Z}$ we have $e^z = 1 + i$. Therefore $\log(1 + i) = \ln(\sqrt{2}) + (\pi/4 + 2n\pi)i$ for $n \in \mathbb{Z}$.

Definition. The principal value of $\log z$ is based on the value of $\arg(z)$ in $(-\pi, \pi]$ which we denote $\Theta$. We define the principal value as

$$\text{Log } z = \ln |z| + i\Theta.$$ 

Note. We have that $\text{Log } z$ is a “single-valued” function (that is, an actual function). The relationship between $\text{Log } z$ and $\log z$ is

$$\log z = \text{Log } z + 2n\pi i \text{ for } n \in \mathbb{Z}.$$ 

(More appropriately, $\log z = \{\text{Log } z + 2n\pi i \mid n \in \mathbb{Z}\}$.)

Example 3.30.2. With $z = 1$ we have $|z| = 1$ and $\arg(z) = \{2n\pi \mid n \in \mathbb{Z}\}$. Therefore $\log 1 = \ln 1 + 2n\pi i = 2n\pi i$ for $n \in \mathbb{Z}$. But $\text{Log } 1 = 0$.

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