Section 3.33. Complex Exponents

Note. In this section we deal with raising a complex number to a complex power. This will be based on logarithms and branches of logarithms and so will lead to the multiple-valued thing again (and the idea of principal values which resolve this issue).

Definition. When $z \neq 0$, we define the “multiple-valued” function $z^c$ for $c \in \mathbb{C}$ as $z^c = e^{c \log z}$.

Note. Since $\log z$ is multiple-valued, then we expect $z^c$ to be multiple valued. The definition is similar to that of the function $a^x$ in terms of $e^x$ in the real setting (see my online notes for Calculus 2 on 7.4. $a^x$ and $\log_a x$; these notes are not based on an “early transcendentals” calculus text and so the theoretical development of logarithms and exponentials is more rigorous). Also notice that this definition of $z^c$ is consistent with the case $c = n \in \mathbb{N}$ as given in Lemma 3.32.C.

Example 3.22.1. Calculate $i^{-2i}$. We have $i^{-2i} = \exp(-2i \log i)$ where

$$\log i = \ln |i| + i \arg(i) = \ln 1 + i \left(\frac{\pi}{2} + 2n\pi\right) = i \left(\frac{\pi}{2} + 2n\pi\right), \ n \in \mathbb{Z}.$$ 

So the multiple-valued result is

$$i^{-2i} = \exp(-2i[i(\pi/2 + 2n\pi)]) = \exp(\pi + 4n\pi) = \exp(\pi(1 + 4n))$$

for $n \in \mathbb{Z}$. Notice that all of the values of $i^{-2i}$ are real and distinct (since the real exponential function is one to one).
Definition. Let log $z$ represent some branch of the logarithm. That is, $\log z = \ln |z| + i\theta$ where $\theta \in \arg(z)$ and $\alpha < \theta < \alpha + 2\pi$. Then a branch of $z^c$ is given by $z^c = e^{c\log z}$. The principal branch of $z^c$ is based on the principal branch of the logarithm (for which we take $\arg(z) \in (-\pi, \pi)$): $z^c = e^{c\text{Log } z}$. The principal branch of $z^c$ gives principal values of $z^c$, which Brown and Churchill denote “P.V. $z^c$.” (Notice that the principal branch of $z^c$ is not defined for nonpositive real numbers).

Note. A common misconception is that the square root function in the real setting is “2-valued.” It is easy to trick a freshman level math student into thinking that $\sqrt{9}$ is $\pm 3$. Of course this is not the case and $\sqrt{9} = 3$. This is because $\sqrt{x}$ is a function (just ask a calculator what $\sqrt{9}$ is). If you want both the positive and negative square roots (which may well be the case in an application) then you must “ask” for both the positive and negative square roots: $\pm \sqrt{9} = \pm 3$. This is related to our situation with branches of $z^c$. With $c = 1/2$, we have that the principal branch of $z^{1/2}$ for $z = 9$ gives

$$\text{P.V. } 9^{1/2} = e^{(1/2)\text{Log } 9} = e^{(1/2)(\ln 9 + i0)} = e^{(1/2)\ln 9} = (e^{\ln 9})^{1/2} = 9^{1/2} = \sqrt{9} = 3.$$ 

Theorem 3.33.A. For any branch of $z^c$, we have $\frac{d}{dz}[z^c] = cz^{c-1}$ where the branch of $z^{c-1}$ is based on the same branch of the logarithm on which $z^c$ is based.

Example. The principal value of $i^i$ is

$$i^i = \exp(i \text{ Log } i) = \exp(i[\ln |i| + i\pi/2]) = \exp(-\pi/2),$$

since $\text{Log } i = \pi/2$. Again, notice that the principal value of $i^i$ is real.
Example 3.33.3. The principal branch of $z^{2/3}$ is

$$\exp((2/3)\log z) = \exp((2/3) \ln |z| + (2/3)i\Theta) = \sqrt{3}|z|^2 \exp(2\Theta i/3)$$

$$= \sqrt{3}|z|^2 (\cos 2\Theta/3 + i \sin 2\Theta/3)$$

where $\Theta$ is the principal argument of $z$ (notice that we must have $\Theta \neq \pi$ and so $z$ cannot be a nonpositive real number when using the principal branch).

Note. The definition of $z^c$ implies that the (multiple-valued) exponential function with base $c$ is $c^z = e^{z \log c}$. Branches and the principal branch of $c^z$ is similarly defined. We find:

$$\frac{d}{dz}[c^z] = c^z \log c.$$

In practice you are unlikely to use any exponential function other than the natural exponential function, $e^z$.

Revised: 1/16/2020