Note. In this section we define the six hyperbolic trigonometric functions and state some identities and properties.

Note. Recall the definition of the real hyperbolic trig functions:
\[
\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.
\]

Just as in the last section, we define new functions of a complex variable in terms of previously constructed functions.

Definition. For any \( z \in \mathbb{C} \) define the hyperbolic cosine and hyperbolic sine as:
\[
\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.
\]

Note 3.35.A. Since \( e^z \) is an entire function, then \( \cosh z \) and \( \sinh z \) are entire functions. Notice that the hyperbolic cosine and sine functions of a complex variable agree with the hyperbolic cosine and sine functions of a real variable when \( z \) is real. Therefore, by Theorem 2.27.A, \( \cosh z \) and \( \sinh z \) are the unique entire functions which agree with \( \cosh x \) and \( \sinh x \). They satisfy the expected differentiation properties (see Exercise 3.35.a; see Exercise 3.39.1 in the 9th edition of the book):
\[
\frac{d}{dx} \cosh z = \sinh z \quad \text{and} \quad \frac{d}{dz} \sinh z = \cosh z.
\]
Note 3.35.B. We can use the definitions of the trigonometric functions in terms of exponential functions to deduce the following (see Exercise 3.39.A):

\[-i \sinh(iz) = \sin z, \cosh(iz) = \cos z,\]
\[-i \sin(iz) = \sinh z, \cos(iz) = \cosh z.\]

The following are confirmed in the exercises:

\[
\begin{align*}
\sinh(-z) &= -\sinh z, \cosh(-z) = \cosh z, \cosh^2 z - \sinh^2 z = 1, \\
\sinh(z_1 + z_2) &= \sinh z_2 \cosh z_2 + \cosh z_1 \sinh z_2, \\
\cosh(z_1 + z_2) &= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2, \\
\sinh z &= \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y, \\
\cosh z &= \cosh(z + iy) = \cosh x \cos y + i \sinh x \sin y, \\
|\sinh z|^2 &= \sinh^2 x + \sin^2 y, \ |\cosh z|^2 = \sinh^2 x + \cos^2 y.
\end{align*}
\]

Note 3.35.C. Since \(\cos z\) and \(\sin z\) have period \(2\pi\), it follows from the identities \(\cosh(iz) = \cos z\) and \(\sinh(iz) = i \sin z\) that \(\cosh z\) and \(\sinh z\) have period \(2\pi i\). It also follows that \(\cosh z = 0\) if and only if \(z = i(\pi/2 + n\pi)\) where \(n \in \mathbb{Z}\), and \(\sinh z = 0\) if and only if \(z = in\pi\) where \(n \in \mathbb{Z}\).

Definition. We define the other four hyperbolic trigonometric functions as:

\[
\begin{align*}
\tanh z &= \frac{\sinh z}{\cosh z}, \ \coth z = \frac{\cosh z}{\sinh z}, \\
\text{sech } z &= \frac{1}{\cosh z}, \ \text{csch } z = \frac{1}{\sinh z}.
\end{align*}
\]
Note 3.35.D. We have the following differentiation formulas:

\[
\frac{d}{dz} \tanh z = \text{sech}^2 z, \quad \frac{d}{dz} \coth z = -\text{csch}^2 z,
\]

\[
\frac{d}{dz} \text{sech} z = -\text{sech} z \tanh z, \quad \frac{d}{dz} \text{csch} z = -\text{csch} z \coth z.
\]

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