Section 4.45. Proof of the Theorem

Note. We now prove the result from the previous section. We start by restating the result.

Theorem 4.44.A. Suppose that a function \( f(z) \) is continuous on a domain \( D \). The following are equivalent:

(a) \( f(z) \) has an antiderivative \( F(z) \) throughout \( D \);

(b) the integrals of \( f(z) \) along contours lying entirely in \( D \) and extending from any fixed point \( z_1 \) to any fixed point \( z_2 \) all have the same value, namely

\[
\int_{z_1}^{z_2} f(z) \, dz = F(z) \bigg|_{z_1}^{z_2} = F(z_2) - F(z_1)
\]

where \( F(z) \) is the antiderivative in statement (a);

(c) the integrals of \( f(z) \) around closed contours lying entirely in \( D \) all have value zero.

Proof. First we show (a) \( \implies \) (b). Suppose \( f(z) \) has an antiderivative \( F(z) \) on the domain \( D \). Let \( C \) be a contour from \( z_1 \) and \( z_2 \) that is smooth, lies in \( D \) and has parametric representation \( z = z(t) \) where \( a \leq t \leq b \). Then by Exercise 4.39.5 we have

\[
\frac{d}{dt} [F(z(t))] = F'(z(t))z'(t) = f(z(t))z'(t) \text{ where } a \leq t \leq b.
\]
So
\[ \int_C f(z) \, dz = \int_Z f(z(t))z'(t) \, dt \text{ by definition (see Section 4.40)} \]
\[ = F(z(t)) \big|_{t=a}^{t=b} = F(z(b)) - F(z(a)) \text{ by Note 4.38.A} \]
\[ = F(z_2) - F(z_1) \text{ since } z_1 = z(a) \text{ and } z_2 = z(b) \]

So (b) holds in the event that \( C \) is smooth. Now a contour is piecewise smooth by definition (see Section 4.39), so for \( C \) any contour that is piecewise smooth, say \( C \) consists of the \( n \) smooth contours \( C_1, C_2, \ldots, C_n \) (with \( C_1 \) a smooth contour from \( z(a) = z_1 \) to \( z_2 \), \( C_2 \) a smooth contour from \( z_2 \) to \( z_3 \), \ldots, and \( C_n \) a smooth contour from \( z_n \) to \( z_{n+1} = z(b) \)), then
\[ \int_C f(z) \, dz = \sum_{k=1}^{n} \int_{C_k} f(z) \, dz \text{ by Note 4.40.C and induction} \]
\[ = \sum_{k=1}^{n} (F(z_{k+1}) - F(z_k)) \text{ by the proof above, since each } C_k \text{ is smooth} \]
\[ = F(b) - F(a). \]

That is, (b) holds.

Next, we show (b) \( \implies \) (c). Suppose that integration of \( f(z) \) is independent of the contour in \( D \) and instead only depends on the endpoints of the contour. Let \( C \) be any closed contour in \( D \) and let \( z_1 \) and \( z_2 \) be two distinct points on \( C \). Form paths \( C_1 \) and \( C_2 \) (see Figure 53).

![Figure 53](image-url)
Since we hypothesize that the values of integrals are independent of contours, then we have \( \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \) or, by Notes 4.40.B and 4.40.C,

\[
0 = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_1-C_2} f(z) \, dz = \int_{C_1-C_2} f(z) \, dz = \int_C f(z) \, dz.
\]

So integrals of \( f(z) \) around closed contours lying entirely in \( D \) all have value zero and (c) holds.

Finally, we show \( (c) \implies (a) \). Suppose integrals of \( f(z) \) around closed contours lying entirely in \( D \) all have value zero. Let \( C_1 \) and \( C_2 \) denote any two contours lying in \( D \) from a point \( z_1 \) to a point \( z_2 \). Then \( C = C_1 - C_2 \) is a closed contour in \( D \) and so by hypothesis,

\[
0 = \int_C f(z) \, dz = \int_{C_1-C_2} f(z) \, dz = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz \text{ by Note 4.40.C}
\]

and so \( \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \) (in fact, we have shown that \( (c) \implies (b) \) here).

Let \( z_0 \in D \) and define function \( F(z) \) as \( F(z) = \int_{z_0}^z f(s) \, ds \) where \( z \in D \). The path independence of integrals shows that \( F \) is well-defined. We now show that \( F'(z) = f(z) \) on \( D \). Let \( z + \Delta z \) be any point distinct from \( z \) and lying in some neighborhood of \( z \) that is small enough to be contained in \( D \) (such a neighborhood exists since \( D \) is hypothesized to be open). Then

\[
F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s) \, ds - \int_{z_0}^z f(s) \, ds = \int_{z}^{z+\Delta z} f(s) \, ds \text{ by Note 4.40.C}.
\]

Since \( \Delta z \) lies in a neighborhood of \( z \) then we see that \( \Delta z \) may be selected as a line segment (see Figure 54).
Section 4.45. Proof of the Theorem

Since \( \int_{z}^{z+\Delta z} ds = \Delta z \) by Exercise 4.42.5 (Exercise 4.46.5 in the 9th edition of the book), we have

\[
\frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(s) \, ds = \frac{f(z)}{\Delta z} \int_{z}^{z+\Delta z} ds = f(z).
\]

So

\[
\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{\int_{z_0}^{z+\Delta z} f(s) \, ds - \int_{z_0}^{z} f(s) \, ds}{\Delta z} = \frac{1}{\Delta z} \left( \int_{z}^{z+\Delta z} f(s) \, ds \right) - \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z) \, ds = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} (f(s) - f(z)) \, ds.
\]

Since \( f \) is continuous at \( z \) by hypothesis, then for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(s) - f(z)| < \varepsilon \) whenever \( |s - z| < \delta \). Consequently, if \( |\Delta z| < \delta \) then

\[
\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| = \left| \frac{1}{\Delta z} \int_{z}^{z+\Delta z} (f(s) - f(z)) \, ds \right| < \frac{1}{|\Delta z|} \varepsilon |\Delta z| \text{ by Theorem 4.43.A}
\]

\[
= \varepsilon.
\]

So by the definition of limit (see Section 2.15),

\[
\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z).
\]

That is, \( F'(z) = f(z) \) and so (a) holds. \( \blacksquare \)