Section 4.54. Maximum Modulus Principle

Note. In this section we present a major result which has many applications. Some of the applications are illustrated in the supplement to this section.

Lemma 4.54.A. Suppose that $|f(z)| \leq |f(z_0)|$ at each point $z$ in some neighborhood $|z - z_0| < \varepsilon$ in which $f$ is analytic. Then $|f(z)|$ has the constant value $f(z_0)$ throughout that neighborhood.

Note. We use Lemma 4.54.A to prove the Maximum Modulus Theorem, but first we elevate equation (2) from the proof of Lemma 4.51.A to the status of a theorem itself.

Theorem 4.54.B. Gauss’s Mean Value Theorem.

Let $f$ be analytic on and inside the positively oriented circle $|z - z_0| = \rho$ centered at $z_0 \in \mathbb{C}$. Then

$$f(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta.$$ 

Note. Recall that, by definition, the average value of a real-valued function of a real variable $f$ on $[a, b]$ is

$$\text{Av}(f) = \frac{1}{b - a} \int_{1}^{b} f(x) \, dx.$$ 

It is this sense that the term “mean” is used in Gauss’s Mean Value Theorem.
**Theorem 4.54.C.** The Maximum Modulus Theorem.

If a function \( f \) is analytic and not constant in a given domain \( D \), then \(|f(z)|\) has no maximum value in \( D \). That is, there is no point \( z_0 \in D \) such that \(|f(z)| \leq |f(z_0)|\) for all points \( z \in D \).

**Theorem 4.54.D.** Maximum Modulus Theorem, Alternative Version.

Suppose that a function \( f \) is continuous on a closed bounded region \( R \) and that it is analytic and not constant in the interior of \( R \). Then the maximum value of \(|f(z)|\) on \( R \), which is always reached (by Theorem 2.18.3) occurs somewhere on the boundary of \( R \) and never in the interior.

**Theorem 4.54.E.** Let \( f \) be continuous on a closed bounded region \( R \), and analytic and not constant on the interior of \( R \). For \( f(z) = u(x, y) + iv(x, y) \), where \( z = x + iy \), function \( u(x, y) \) attains its maximum value in \( R \) on the boundary of \( R \) and not in the interior.

**Note.** The proof of the following is to be given in Exercise 4.54.3.

**Corollary 4.54.F.** The Minimum Modulus Theorem.

Let a function \( f \) be continuous on a closed bounded region \( R \), and let it be analytic and not constant throughout the interior of \( R \). Assuming that \( f(z) \neq 0 \) anywhere in \( R \), then \(|f(z)|\) has a minimum value \( m \) in \( R \) and the minimum is attained at some boundary point of \( R \) and is never attained at an interior point of \( R \).
Note. Another version of the Maximum Modulus Theorem is the following, a proof of which is given in my online class notes for Complex Analysis (MATH 5510-20) on Section VI.1. The Maximum Principle.

Theorem 4.54.G. Maximum Modulus Theorem for Unbounded Domains (Simplified 1).
Let $r > 0$ and suppose $f$ is analytic for $|z| > r$, continuous on $|z| = r$, bounded by $M$ on $|z| = r$, and $\lim_{|z| \to \infty} |f(z)| \leq M$. Then $|f(z)| \leq M$ on $|z| \geq r$.

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