Section 5.59. Examples

Note. We now apply Taylor’s Theorem (Theorem 5.57.A) to find series representations for several functions. In each example, we must be aware of parameter $R_0$.

**Example 5.59.1.** The function $f(z) = e^z$ is entire (since $f'(z) = e^z$ for all $z \in \mathbb{C}$ by Exercise 2.22.A) so by Taylor’s Theorem, $f(z)$ has a Maclaurin series $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ valid for all $z \in \mathbb{C}$ (that is, $R_0 = \infty$). Here, $f^{(n)}(z) = e^z$ for $n \in \mathbb{N} \cup \{0\}$ and so $f^{(n)}(0) = 1$ for all $n$. So

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \text{ for } |z| < \infty.$$ 

We can find a series for the entire function $z^2e^{3z}$ by replacing $z$ with $3z$ in the above series (and noticing that $|3z| < \infty$ is equivalent to $|z| < \infty$) to get first that

$$e^{3z} = \sum_{n=0}^{\infty} \frac{1}{n!} (3z)^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^n \text{ for } |z| < \infty.$$ 

Next we multiply both sides by $z^2$ and distribute on the right-hand side (this can by justified pointwise by Exercise 5.56.7) to get

$$z^2e^{3z} = z^2 \sum_{n=0}^{\infty} \frac{3^n}{n!} z^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \text{ for } |z| < \infty.$$ 

We can similarly find a series for $e^{-z^2}$ as

$$e^{-z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{2n} \text{ for } |z| < \infty.$$
With \( z = x \) real, we have \( e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \) for \( |x| < \infty \), which is a useful series in statistics (it is related to the normal distribution and can be used to calculate the numerical values in a Z-table).

**Example 5.59.2.** In Section 3.34, “Trigonometric Functions,” we defined \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \). Since we now have a Maclaurin series for \( e^z \), we can present such a series for \( \sin z \):

\[
\sin z = \frac{1}{2i} \left( \sum_{n=0}^{\infty} \frac{1}{n!} (iz)^n - \sum_{n=0}^{\infty} \frac{1}{n!} (-iz)^n \right) \text{ for } |z| < \infty
\]

\[
= \frac{1}{2i} \left( \sum_{n=0}^{\infty} \frac{i^n}{n!} z^n - \sum_{n=0}^{\infty} \frac{(-1)^n i^n}{n!} z^n \right)
\]

\[
= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(1 - (-1)^n) i^n}{n!} z^n \text{ (this can be justified pointwise by Exercise 5.56.8)}
\]

\[
= \frac{1}{2i} \sum_{n=0, n \text{ odd}}^{\infty} \frac{(1 - (-1)^n) i^n}{n!} z^n \text{ since } (1) - (-1)^n = 0 \text{ for } n \text{ even}
\]

\[
= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(1 - (-1)^{2n+1}) i^{2n+1}}{(2n+1)!} z^{2n+1} \text{ replacing odd } n \text{ above}
\]

\[
= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{2(-1)^n i^{2n+1}}{(2n+1)!} z^{2n+1} \text{ since } 1 - (-1)^{2n+1} = 2 \text{ and } i^{2n+1} = (i^2)^n i = (-1)^n i
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \text{ for } |z| < \infty.
\]

So

\[
\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \text{ for } |z| < \infty.
\]

In Theorem 5.65.2 we’ll see that a power series can be differentiated term-by-term.
so that
\[
\cos z = \frac{d}{dz} \sin z = \frac{d}{dz} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \text{ for } |z| < \infty.
\]

**Example 5.59.4.** Consider \( f(z) = \frac{1}{1-z} = (1-z)^{-1} \). We have

\[
f^{(n)}(z) = n!(1-z)^{-(n+1)} = \frac{n!}{(1-z)^{n+1}} \text{ and } f^{(n)}(0) = n! \text{ for } n \in \mathbb{N} \cup \{0\}.
\]

Now \( f(z) \) is not defined at \( z = 1 \) so that the Maclaurin series for \( f(z) \) can have radius of convergence \( R_0 \) at most 1. The Maclaurin series is

\[
\frac{1}{1-z} = f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n,
\]

and as we saw in an example from Section 5.56, “Convergence of Series,” this series in fact converges for \( |z| < R_0 = 1 \). So we have

\[
\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ for } |z| < 1.
\]

If we replace \( z \) with \( 1 - z \) then we get the Taylor series

\[
\frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (1-z)^n \text{ for } |z - 1| < 1.
\]

**Example 5.59.5.** We seek a series representation of the rational function

\[
f(z) = \frac{1 + 2z^2}{z^3 + z^5} = \frac{1}{z^3} \left( \frac{2(1 + z^2) - 1}{1 + z^2} \right) = \frac{1}{z^3} \left( 2 - \frac{1}{1 + z^2} \right).
\]
First, we use Example 5.59.4 to get $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$, and replacing $z$ with $-z^2$ we see that

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

for $|z| < 1$ and so (since $1/z^3$ is not defined at $z = 0$):

$$f(z) = \frac{1}{z^3} \left( 2 - \frac{1}{1+z^2} \right) = \frac{1}{z^3} \left( 2 - \sum_{n=0}^{\infty} (-1)^n z^{2n} \right)$$

for $0 < |z| < 1$.

This is a series representation of $f$, but it is not a Taylor series since it involves some negative powers of $z$. We’ll see in the next section that such a series is called a “Laurent series.”

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