Section 5.62. Examples

Note. We now find Laurent series for several functions. In each example, we are careful to give a set on which the series is valid. However, we do not compute coefficients using integrals as stated in Laurent’s Theorem (Theorem 5.60.1).

Example 5.62.1. Since \( e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \) for \( |z| < \infty \), then replacing \( z \) with \( 1/z \) we have
\[
e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} \text{ for } 0 < |z| < \infty.
\]

Based on Laurent’s Theorem (Theorem 5.60.1), we see that \( b_1 = \frac{1}{2\pi i} \int_C e^{1/z} \, dz \) where \( C \) is any positively oriented simple closed contour around \( z_0 = 0 \). Since \( b_1 = 1 \) here, then \( \int_C e^{1/z} \, dz = 2\pi i \). So if we have a Laurent series for \( f(z) \), then we can use it and Laurent’s Theorem to evaluate certain integrals. This is explained in more detail in Chapters 6 and 7 (“Residues and Poles” and “Applications of Residues,” respectively).

Example 5.62.2. The function \( f(z) = \frac{1}{(z - i)^2} \) is already in the form of a Laurent series where \( z_0 = i \). With
\[
\frac{1}{(z - i)^2} = \sum_{n=-\infty}^{\infty} c_n (z - i)^n \text{ where } 0 < |z - i| < \infty
\]
we have \( c_{-2} = 1 \) and \( c_n = 0 \) for \( n \in \mathbb{Z} \setminus \{-2\} \). By Laurent’s Theorem (see Note 5.60.A)
\[
c_n = \frac{1}{2\pi i} \int_C \frac{f(z) \, dz}{(z - z_0)^{n+1}} \text{ for } n \in \mathbb{Z}
\]
and $C$ any positively oriented simple closed contour around $z_0 = i$ lying in that domain $0 < |z - i| < \infty$. Therefore

$$\int_{C} \frac{dz}{(z - i)^{n+3}} = \begin{cases} 0 & \text{for } n \in \mathbb{Z} \setminus \{-2\} \\ 2\pi i & \text{for } n = -2. \end{cases}$$

**Examples 5.62.3 and 5.62.4.** Consider the function

$$f(z) = \frac{-1}{(z - 1)(z - 2)} = \frac{1}{z - 1} - \frac{1}{z - 2}.$$  

This has two singular points, $z = 1$ and $z = 2$. So $f(z)$ is analytic in $\mathbb{C} \setminus \{1, 2\}$. In particular, $f$ is analytic in the domains $|z| < 1$, $1 < |z| < 2$, and $2 < |z| < \infty$, which we denote $D_1$, $D_2$, and $D_3$ respectively (see Figure 78).

![Figure 78](image)

As seen above, we have $\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$. Replacing $z$ with $z/2$ gives

$$\frac{1}{1 - z/2} = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{2^n} \text{ for } |z| < 2.$$  

So in domain $D_1$ we have

$$f(z) = \frac{1}{z - 1} - \frac{1}{z - 2} = -\frac{1}{1 - z} + \frac{1}{2 - z} = -\frac{1}{1 - z} + \frac{1}{2} \left(\frac{1}{1 - z/2}\right).$$
\[- \sum_{n=0}^{\infty} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1}} - 1 \right) z^n \text{ for } |z| < 1.\]

So \( f(z) \) actually has a Taylor series representation on \( D_1 \).

Next, we look for a series representation on \( D_2 \). We need to modify our version of \( f \) so that we can get a series representation valid outside of \( |z| \leq 1 \). So we write

\[
f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \frac{1}{1-1/z} + \frac{1}{2} \frac{1}{1-z/2}.
\]

Replacing \( z \) with \( 1/z \) in the series for \( \frac{1}{1-z} \) we get

\[
\frac{1}{1-1/z} = \sum_{n=0}^{\infty} \left( \frac{1}{z} \right)^n \text{ for } |1/z| < 1 \text{ (or equivalently } |z| > 1).\]

So in domain \( D_2 \) we have

\[
f(z) = \frac{1}{z} \frac{1}{1-1/z} + \frac{1}{2} \frac{1}{1-z/2}.
\]

\[
= \frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{1}{z} \right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} \text{ for } 1 < |z| < 2
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \text{ for } 1 < |z| < 2.
\]

Finally, we look for a series representation on \( D_3 \). This time we write

\[
f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \frac{1}{1-1/z} - \frac{1}{z} \frac{1}{1-2/z}.
\]

From above,

\[
\frac{1}{z} \frac{1}{1-1/z} = \frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{1}{z} \right)^n = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \text{ for } |z| > 1.
\]

Similarly, replacing \( z \) with \( 2/z \) in the series for \( \frac{1}{1-z} \) we get

\[
\frac{1}{1-2/z} = \sum_{n=0}^{\infty} \left( \frac{2}{z} \right)^n = \sum_{n=0}^{\infty} \frac{2^n}{z^n} \text{ for } \left| \frac{2}{z} \right| < 1 \text{ (or equivalently } |z| > 2).\]
So in domain $D_3$ we have

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \frac{1}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n} = \sum_{n=0}^{\infty} \frac{1 - 2^n}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n}$$

for $|z| > 2$.

So in these examples we see that the same function may have different Laurent series centered at $z_0$ which are valid in different regions.