Section 6.72. The Three Types of Isolated Singular Points

Note. In this section we use the Laurent series of a function centered at an isolated singular point to classify the singular point into one of three categories.

Definition. If for function $f$ there is $R_1 > 0$ such that $f$ is analytic for $R_1 < |z| < \infty$ then $f$ has an isolated singularity point at $z_0 = \infty$.

Note. Suppose $f$ has an isolated singular point at $z_0$; that is, $f$ is analytic for $0 < |z - z_0| < R_2$ but $f$ is not analytic at $z_0$, then by Theorem 5.60.1, “Laurent’s Theorem,”

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n \text{ for } 0 < |z - z_0| < R_2.$$  

Definition. Let $f$ have an isolated singular point at $z = z_0$ and let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m}$$

for $0 < |z - z_0| < R_2$ (that is, $c_0 = 0$ for $n < -m$). Then $z_0$ is a pole of order $m$ for $f$. If $z_0$ is a pole of order 1 then it is a simple pole of $f$.

Definition. Let $f$ have an isolated singular point at $z = z_0$ and let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n = \sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ for } 0 < |z - z_0| < R_2$$

(that is, $c_n = 0$ for $n < 0$). Then $z_0$ is a removable singular point of $f$. 
Note 6.72.A. If $f$ has a removable singular point at $z_0$ then we can define $g(z)$ on $|z - z_0| < R_2$ as

$$g(z) = \begin{cases} 
  f(z) & \text{for } 0 < |z - z_0| < R_2 \\
  z_0 & \text{for } z = z_0.
\end{cases}$$

Then $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and so $g$ is analytic on $|z - z_0| < R_2$. Notice that this means $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z) = g(z_0)$. In this way, the singularity of $f$ at $z = z_0$ has been “removed.” An example of this is given by the functions $f(z) = (z^2 - 1)/(z - 1)$ and $g(z) = z + 1$.

**Definition.** Let $f$ have an isolated singular point at $z = z_0$. If $z_0$ is neither a removable singular point nor a pole then $z_0$ is an *essential singular point*.

**Note.** If $f$ has an essential singular point at $z = z_0$ then the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$ for $0 < |z - z_0| < R_2$ must have infinitely many nonzero $c_n$ for $n < 0$.

**Example.** (Exercise 6.72.1(c)) Consider $f(z) = (\sin z)/z$. Since $\sin z = (e^{-z} - e^{-iz})/2$ (see Section 3.34, “Trigonometric Functions”) we have

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n + 1)!}$$

for all $z \in \mathbb{C}$

and so

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n + 1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n + 1)!}$$

for $0 < |z| < \infty$.
and so \((\sin z)/z\) has a removable singularity at \(z = 0\). Notice
\[
\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \left( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \right) = \frac{(-1)^0}{(2(0)+1)!} = 1.
\]

**Example 3.** Consider \(f(z) = \frac{\sinh z}{z^4}\). Since \(\sinh z = (e^z - e^{-z})/2\) (see Section 3.35, “Hyperbolic Functions”) we have \(\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\) for all \(z \in \mathbb{C}\) and so
\[
\frac{\sinh z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n-3}}{(2n+1)!}
\]
\[
= \frac{1}{z^3} + \frac{1}{6z} + \sum_{n=2}^{\infty} \frac{z^{2n-3}}{(2n+1)!} = \frac{1}{z^3} + \frac{1}{6z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+5)!}
\]
and so \(f\) has a pole of order \(m = 3\) at \(z_0 = 0\).

**Example 5.** Consider \(f(z) = \exp(1/z)\). Since \(\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}\) for all \(z \in \mathbb{C}\) then
\[
\exp(1/z) = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{z^n n!} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \quad \text{for } 0 < |z| < \infty
\]
and so \(f\) has an essential singularity at \(z_0 = 0\).

**Note.** Brown and Churchill now mention (without a formal statement) “Picard’s Theorem.” This claims that if \(f\) has an essential singularity at \(z = z_0\) then for all \(\varepsilon > 0\) such that \(f\) is analytic on \(0 < |z - z_0| < \varepsilon\), function \(f\) assumes every value \(c \in \mathbb{C}\) an infinite number of times, with one possible exception for the value \(c\) (for \(f(z) = \exp(1/z)\) the exceptional value is \(c = 0\)). For details, see Section XII.1,
"The Great Picard Theorem in John Conway’s *Functions of One Complex Variable I*, 2nd Edition [Springer-Verlag, 1978]. This is the last section of this graduate-level text, reflecting the background needed to prove the result.

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