Section 6.76. Zeros and Poles

Note. We now relate the zeros of order \( m \) for an analytic function to the poles of order \( m \) in the reciprocal of the analytic function.

Theorem 6.76.1. Suppose that

(a) two functions \( p \) and \( q \) are analytic at a point \( z_0 \), and

(b) \( p(z_0) \neq 0 \) and \( q \) has a zero of order \( m \) at \( z_0 \).

Then the quotient \( p(z)/q(z) \) has a pole of order \( m \) at \( z_0 \).

Example 6.76.1. Let \( p(z) = 1 \) and \( z(z) = z(e^z - 1) \). Then \( p \) and \( q \) are entire and by Example 6.75.2, \( q \) has a zero of order \( m = 2 \) at \( z_0 = 0 \). So by Theorem 6.76.a, \( \frac{p(z)}{q(z)} = \frac{1}{z(e^z - 1)} \) has a pole of order 2 at \( z_0 = 0 \) (as we saw in Example 6.74.5).

Theorem 6.76.2. Let the functions \( p \) and \( q \) be analytic at \( z_0 \). If \( p(z_0) \neq 0 \), \( q(z_0) = 0 \), and \( q'(z_0) = 0 \) (that is, \( q \) has a zero of multiplicity one at \( z_0 \)) then \( z_0 \) is a simple pole of \( p/q \) and \( \text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} \).
**Example 6.76.2.** Consider \( f(z) = \cos z / \sin z = \cot z \). With \( p(z) = \cos z \) and \( q(z) = \sin z \). Since \( a(z) = \sin z = 0 \) for \( z = n\pi \) where \( n \in \mathbb{Z} \), \( q'(z) = \cos z \), \( q'(n\pi) = (-1)^n \neq 0 \), and \( p(n\pi) = (-1)^n \neq 0 \) then by Theorem 6.76.2, \( f \) has a simple pole at each \( n\pi \) where \( n \in \mathbb{Z} \) and

\[
\text{Res}_{z=n\pi} f(z) = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1.
\]

**Example 6.76.4.** Consider \( f(z) = \frac{z}{z^4 + 4} \). Let \( p(z) = z \) and \( q(z) = z^4 + 4 \). Then for \( z_0 = \sqrt{2} \exp(i\pi/4) = 1 + i \) and \( p(z_0) = z_0 \neq 0 \), \( q(z_0) = 0 \), and \( q'(z_0) = 4z_0^3 \neq 0 \). So by Theorem 6.76.2, \( f \) has a simple pole at \( z_0 \), and the residue at \( z_0 \) is

\[
\text{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{4(2\exp(i\pi/2))} = \frac{1}{8i} = -\frac{i}{8}.
\]

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