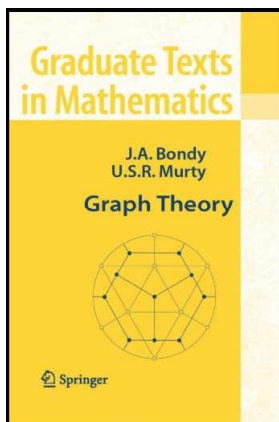


Graph Theory

Chapter 1. Graphs

1.1. Graphs and Their Representations—Proofs of Theorems



Theorem 1.1

Theorem 1.1. For any graph G , $\sum_{v \in V} d(v) = 2m$ where $m = |E|$.

Proof. Consider the incidence matrix \mathbf{M} of G . For given $v \in V$, entry m_{ve} is the number of times edge e is incident with vertex v . So as e ranges over set E , we have $\sum_{e \in E} m_{ve} = d(v)$. Now the row of \mathbf{M} corresponding to vertex v has exactly the entries m_{ve} where e ranges over edge set E . So the sum of the entries in this row is also $d(v)$. Therefore $\sum_{v \in V} d(v)$ is the sum of all entries in \mathbf{M} . Now the sum of the entries of any column in \mathbf{M} is 2, since the column entries (of 0, 1, or 2) reflect the ends of the edge corresponding to that column, and each edge has 2 ends. The total number of columns is $m = |E|$, so the sum of all entries in \mathbf{M} is $2m$. That is, $\sum_{v \in V} d(v) = 2m$, as claimed. \square

Corollary 1.2

Corollary 1.2. In any graph, the number of vertices of odd degree is even.

Proof. Let $V_1 = \{v \in V \mid d(v) \text{ is odd}\}$ and let $V_2 = \{v \in V \mid d(v) \text{ is even}\}$. Then

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = \sum_{v \in V} d(v).$$

Now $\sum_{v \in V_2} d(v)$ is even since each such $d(v)$ is even and, by Theorem 1.1, $\sum_{v \in V} d(v)$ is even. Therefore $\sum_{v \in V_1} d(v)$ must also be even. Since each such $d(v)$ is odd then $|V_1|$ must be even. That is, the number of vertices of odd degree is even, as claimed. \square

(This proof is from Bondy and Murty's *Graph Theory with Applications* (North Holland, 1976.)

Proposition 1.3

Proposition 1.3. Let $G[X, Y]$ be a bipartite graph without isolated vertices such that $d(x) \geq d(y)$ for all $x \in X$ and $y \in Y$ such that $\psi_G(e) = \{x, y\} = xy$ for some $e \in E$ (we abbreviate $\psi_G(e) = \{x, y\} = xy$ for some $e \in E$ simply as " $xy \in E$ "). Then $|X| \leq |Y|$, with equality if and only if $d(x) = d(y)$ for all $xy \in E$.

Proof. Consider the bipartite adjacency matrix \mathbf{B} for $G[X, Y]$. Create matrix $\tilde{\mathbf{B}}$ by dividing the row of \mathbf{B} corresponding to vertex x by $d(x)$, and do so for each $x \in X$ (notice that $d(x) > 0$ for each $x \in X$ since G has no isolated vertices by hypothesis). Since the sum of the entries in the row of \mathbf{B} corresponding to vertex x is $d(x)$ (because \mathbf{B} is an adjacency matrix) then the sum of the entries of the row of $\tilde{\mathbf{B}}$ corresponding to vertex x is 1 and the sum of all entries in $\tilde{\mathbf{B}}$ is $|X|(1) = |X|$.

Proposition 1.3 (continued 1)

Proof (continued). The sum of the entries in the column of $\tilde{\mathbf{B}}$ corresponding to vertex $y \in Y$ is $\sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$ (that is, the sum of the reciprocals of the degrees of the vertices in X which are adjacent to y). So the sum of all entries in $\tilde{\mathbf{B}}$ is also $\sum_{y \in Y} \sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$. Since we have summed the entries of $\tilde{\mathbf{B}}$ in two ways, we have $|X| = \sum_{y \in Y} \sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$. Now summing over all $y \in Y$ and $(x \in X \text{ such that } xy \in E)$ is equivalent to summing over all $(x \in X \text{ and } y \in Y)$ and $xy \in E$.

Proposition 1.3 (continued 2)

Proof (continued). So

$$\begin{aligned} |X| &= \sum_{y \in Y} \sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)} = \sum_{x \in X, y \in Y} \sum_{xy \in E} \frac{1}{d(x)} \\ &\leq \sum_{x \in X, y \in Y} \sum_{xy \in E} \frac{1}{d(y)} \text{ since } d(x) \geq d(y) \text{ for all } xy \in E \\ &= \sum_{x \in X} \sum_{\{y \in Y \mid xy \in E\}} \frac{1}{d(y)} \text{ as argued above} \\ &\quad \text{(with sets } X \text{ and } Y \text{ interchanged here)} \\ &= |Y| \text{ since the sum of the entries in the columns corresponding} \\ &\quad \text{to vertex } y \text{ sum to } d(y) \text{ in } \mathbf{B} \text{ and to } 1 \text{ in } \tilde{\mathbf{B}} \\ &\quad \text{(similar to the rows, as described above).} \end{aligned}$$

So $|X| \leq |Y|$. The only way to have equality is when the inequality above reduces to an equality and this requires that $d(x) = d(y)$ for all $xy \in E$, as claimed. \square