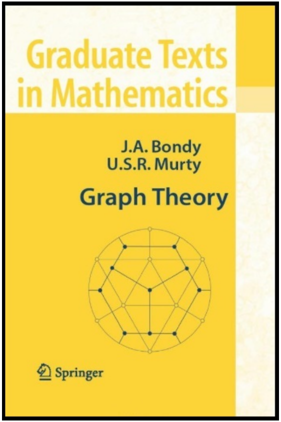
# Graph Theory

## Chapter 1. Graphs

1.1. Graphs and Their Representations—Proofs of Theorems



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# Theorem 1.1

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**Proof.** Consider the incidence matrix $\mathbf{M}$ of $G$. For given $v \in V$, entry $m_{ve}$ is the number of times edge $e$ is incident with vertex $v$. So as $e$ ranges over set $E$, we have $\sum_{e \in E} m_{ve} = d(v)$. Now the row of $\mathbf{M}$ corresponding to vertex $v$ has exactly the entries $m_{ve}$ where $e$ ranges over edge set $E$. So the sum of the entries in this row is also $d(v)$. Therefore $\sum_{v \in V} d(v)$ is the sum of all entries in $\mathbf{M}$.

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# Corollary 1.2

**Corollary 1.2.** In any graph, the number of vertices of odd degree is even.

**Proof.** Let $V_1 = \{v \in V \mid d(v) \text{ is odd}\}$ and let $V_2 = \{v \in V \mid d(v) \text{ is even}\}$. Then

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = \sum_{v \in V} d(v).$$

Now $\sum_{v \in V_2} d(v)$ is even since each such $d(v)$ is even and, by Theorem 1.1, $\sum_{v \in V} d(v)$ is even. Therefore $\sum_{v \in V_1} d(v)$ must also be even. Since each such $d(v)$ is odd then $|V_1|$ must be even. That is, the number of vertices of odd degree is even, as claimed. □

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# Proposition 1.3

**Proposition 1.3.** Let $G[X, Y]$ be a bipartite graph without isolated vertices such that $d(x) \geq d(y)$ for all $x \in X$ and $y \in Y$ such that $\psi_G(e) = \{x, y\} = xy$ for some $e \in E$ (we abbreviate $\psi_G(e) = \{x, y\} = xy$ for some $e \in E$ simply as "$xy \in E$"). Then $|X| \leq |Y|$, with equality if and only if $d(x) = d(y)$ for all $xy \in E$.

**Proof.** Consider the bipartite adjacency matrix $\mathbf{B}$ for $G[X, Y]$. Create matrix $\tilde{\mathbf{B}}$ by dividing the row of $\mathbf{B}$ corresponding to vertex $x$ by $d(x)$, and do so for each $x \in X$ (notice that $d(x) > 0$ for each $x \in X$ since $G$ has no isolated vertices by hypothesis). Since the sum of the entries in the row of $\mathbf{B}$ corresponding to vertex $x$ is $d(x)$ (because $\mathbf{B}$ is an *adjacency* matrix) then the sum of the entries of the row of $\tilde{\mathbf{B}}$ corresponding to vertex $x$ is 1 and the sum of all entries in $\tilde{\mathbf{B}}$ is $|X|(1) = |X|$.

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# Proposition 1.3 (continued 1)

**Proof (continued).** The sum of the entries in the column of $\tilde{\mathbf{B}}$ corresponding to vertex $y \in Y$ is $\displaystyle\sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$ (that is, the sum of the reciprocals of the degrees of the vertices in $X$ which are adjacent to $y$). So the sum of all entries in $\tilde{\mathbf{B}}$ is also $\displaystyle\sum_{y \in Y} \sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$. Since we have summed the entries of $\tilde{\mathbf{B}}$ in two ways, we have $\displaystyle |X| = \sum_{y \in Y} \sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$. Now summing over all $y \in Y$ and ($x \in X$ such that $xy \in E$) is equivalent to summing over all ($x \in X$ and $y \in Y$) and $xy \in E$.

# Proposition 1.3 (continued 1)

**Proof (continued).** The sum of the entries in the column of $\tilde{\mathbf{B}}$ corresponding to vertex $y \in Y$ is $\displaystyle\sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$ (that is, the sum of the reciprocals of the degrees of the vertices in $X$ which are adjacent to $y$). So the sum of all entries in $\tilde{\mathbf{B}}$ is also $\displaystyle\sum_{y \in Y} \sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$. Since we have summed the entries of $\tilde{\mathbf{B}}$ in two ways, we have $\displaystyle |X| = \sum_{y \in Y} \sum_{\{x \in X \mid xy \in E\}} \frac{1}{d(x)}$. Now summing over all $y \in Y$ and ($x \in X$ such that $xy \in E$) is equivalent to summing over all ($x \in X$ and $y \in Y$) and $xy \in E$.

# Proposition 1.3 (continued 2)

**Proof (continued).** So

$$|X| = \sum_{y \in Y} \sum_{\{x \in X | xy \in E\}} \frac{1}{d(x)} = \sum_{x \in X, y \in Y} \sum_{xy \in E} \frac{1}{d(x)}$$

$$\leq \sum_{x \in X, y \in Y} \sum_{xy \in E} \frac{1}{d(y)} \text{ since } d(x) \geq d(y) \text{ for all } xy \in E$$

$$= \sum_{x \in X} \sum_{\{y \in Y | xy \in E\}} \frac{1}{d(y)} \text{ as argued above}$$

(with sets $X$ and $Y$ interchanged here)

$= |Y|$ since the sum of the entries in the columns corresponding to vertex $y$ sum to $d(y)$ in $\mathbf{B}$ and to 1 in $\tilde{\mathbf{B}}$ (similar to the rows, as described above).

So $|X| \leq |Y|$. The only way to have equality is when the inequality above reduces to an equality and this requires that $d(x) = d(y)$ for all $xy \in E$, as claimed. □

# Proposition 1.3 (continued 2)

**Proof (continued).** So

$$\begin{aligned} |X| &= \sum_{y \in Y} \sum_{\{x \in X | xy \in E\}} \frac{1}{d(x)} = \sum_{x \in X, y \in Y} \sum_{xy \in E} \frac{1}{d(x)} \\ &\leq \sum_{x \in X, y \in Y} \sum_{xy \in E} \frac{1}{d(y)} \text{ since } d(x) \geq d(y) \text{ for all } xy \in E \\ &= \sum_{x \in X} \sum_{\{y \in Y | xy \in E\}} \frac{1}{d(y)} \text{ as argued above} \\ &\quad \text{(with sets } X \text{ and } Y \text{ interchanged here)} \\ &= |Y| \text{ since the sum of the entries in the columns corresponding} \\ &\quad \text{to vertex } y \text{ sum to } d(y) \text{ in } \mathbf{B} \text{ and to 1 in } \tilde{\mathbf{B}} \\ &\quad \text{(similar to the rows, as described above).} \end{aligned}$$

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