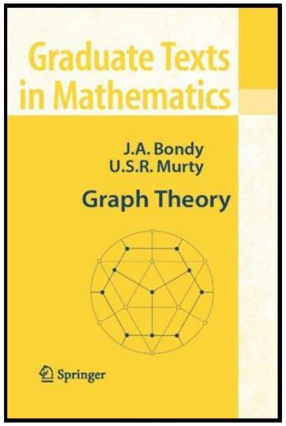
Graph Theory

Chapter 10. Planar Graphs

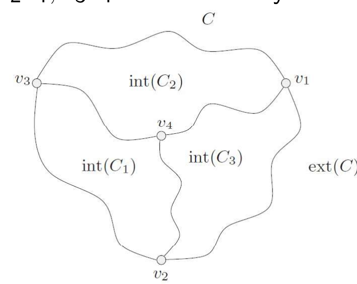
10.1. Plane and Planar Graphs—Proofs of Theorems



Theorem 10.2

Theorem 10.2. K_5 is nonplanar.

Proof. ASSUME $G = K_5$ is planar. Let $\tilde{G}$ be a planar embedding of K_5, with points v_1, v_2, v_3, v_4, v_5. Since K_5 is complete, any two points of $\tilde{G}$ are joined by a line. The cycle $C = v_1 v_2 v_3 v_1$ is a simple closed curve in $\mathbb{R}^2$, and the point v_4 must lie either in $\text{int}(C)$ or in $\text{ext}(C)$. Without loss of generality we can suppose $v_4 \in \text{int}(C)$ (or else we can permute the roles of v_1, v_2, v_3, v_4 and get a different cycle C and a different interior point). Then the edges $v_1 v_4, v_2 v_4, v_3 v_4$ all lie entirely in $\text{int}(C)$ (apart from their end points v_1, v_2, v_3):



Theorem 10.2 (continued)

Theorem 10.2. K_5 is nonplanar.

Proof (continued). Consider the cycles $C_1 = v_2 v_3 v_4 v_2$, $C_2 = v_3 v_1 v_4 v_3$, and $C_3 = v_1 v_2 v_4 v_1$. We have $v_i \in \text{ext}(C_i)$ for $i = 1, 2, 3$ (as seen in Figure 10.3 above). Now $v_i v_5 \in E(\tilde{G})$ for $i = 1, 2, 3$, so by the Jordan Curve Theorem we have $v_5 \in \text{ext}(C_i)$ for $i = 1, 2, 3$ (for example, if $v_5 \in \text{int}(C_1)$ then the line joining v_5 and $v_1 \in \text{ext}(C_1)$ must intersect cycle C_1, contradicting the planarity of K_5). So $v_5 \in \text{ext}(C)$ as well. But then the line joining v_4 and v_5 crosses C by the Jordan Curve Theorem, CONTRADICTING the planarity of K_5. Hence the assumption that K_5 is planar is false and therefore K_5 is nonplanar, as claimed. □