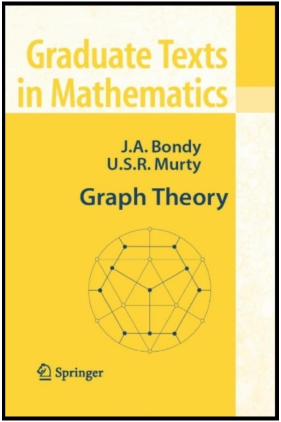
Graph Theory

Chapter 10. Planar Graphs

10.1. Plane and Planar Graphs—Proofs of Theorems



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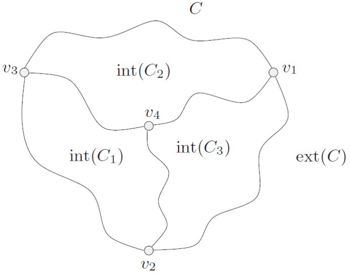

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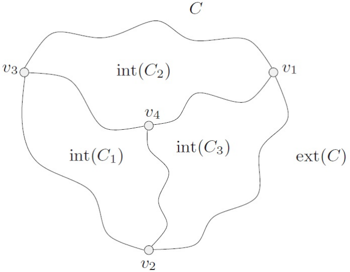

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