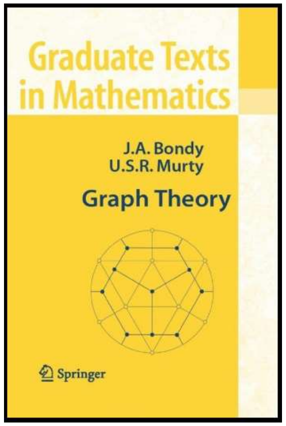
Graph Theory

Chapter 10. Planar Graphs

10.2. Duality—Proofs of Theorems



Theorem 10.5

Theorem 10.5. Let G be a planar graph and let f be a face in some planar embedding of G. Then G admits a planar embedding whose out face has the same boundary as f.

Proof. (This argument is based somewhat on geometric intuition.) Consider an embedding $\tilde{G}$ of G on the sphere, which exists by Theorem 10.4 since G is hypothesized to be planar. Let $\tilde{f}$ be the face of $\tilde{G}$ corresponding to (plane) face f. Let z be a point on the sphere in the interior of $\tilde{f}$. Let $\pi(\tilde{G})$ be the image of $\tilde{G}$ under stereographic projection from z (so z itself is mapped to ∞ in the extended plane). Then $\pi(\tilde{G})$ is a planar embedding (also, arguably, by Theorem 10.4) and face $\tilde{f}$ of $\tilde{G}$ is mapped to the outer face of $\pi(\tilde{G})$ so that the boundary of $\pi(\tilde{f})$ is the same as the boundary of f (in terms of the edges and vertices it contains), as claimed. □

Theorem 10.7

Theorem 10.7. In a nonseparable plane graph other than K_1 or K_2, each face is bounded by a cycle.

Proof. Let G be a nonseparable plane graph, other than K_1 or K_2. Then by Theorem 5.8 of Section 5.3. Ear Decompositions, G has an ear decomposition, say $G_0, G_1, \ldots, G_k$, where G_0 is a cycle, $G_k = G$, and for $0 \le i \le k-1$, $G_{i+1} = G_i \cup P_i$ is a nonseparable plane subgraph of G where P_i is an ear of G_i. Since G_0 is a cycle, the two faces of G_0 (the "interior" and "exterior" of the cycle by the Jordan Curve Theorem), then the two faces of G_0 are bounded by cycles. We show the claim using induction. Suppose that all faces of G_i are bounded by cycles where $i \ge 0$ is given.

Theorem 10.7 (continued)

Theorem 10.7. In a nonseparable plane graph other than K_1 or K_2, each face is bounded by a cycle.

Proof (continued). Since G is a plane graph then G_{i+1} is a plane graph and the ear P_i of G_i is contained (except for its end vertices) in some face f of G_i. Each face of G_i other than f is a face of G_{i+1} and so (by the induction hypothesis) is bounded by a cycle. Now the face f of G_i (we can view this as subdividing face f of G_i by adding a single edge and then subdividing that edge by adding additional vertices). Each of these two faces are also bound by cycles and so the result holds for $i+1$ and G_{i+1}. By induction the claim holds for all $0 \le i \le k$ and so holds for $G_k = G$, as claimed. □

Corollary 10.8

Corollary 10.8. In a loopless 3-connected plane graph, the neighbors of any vertex lie on a common cycle.

Proof. Let G be a loopless 3-connected plane graph and let v be any vertex of G. Since G is 3-connected then $G - v$ is 2-connected and so is nonseparable. So by Theorem 10.7 each face of $G - v$ is bounded by a cycle. With f as the face of $G - v$ which contains vertex v, then the neighbors of v must lie on the cycle which bounds face f in graph $G - v$. Since v is an arbitrary vertex of G, then the result holds. □

Proposition 10.9

Proposition 10.9. A dual G^* of a plane graph G is connected.

"Proof." Let G be a plane graph and G^* a plane dual of G. Consider two vertices f^* and g^* of G^*. The faces of G partition the plane minus G, $\mathbb{R}^2 \setminus G$, into a finite number of pieces so we can start with face f and "move" to face f_1 (along edge e^* in G^*, where edge e of G separates faces f and f_1) and to face $f_2, \ldots,$ and to face g (this is the weak part of the proof). This sequence of faces of G and edges of G^* determines a walk in G^* from vertex f^* to vertex g^* of G^*. Therefore, since f^* and g^* are arbitrary vertices of G^*, by Exercises 3.1.4 and 3.1.1, G^* is connected. □

Theorem 10.10

Theorem 10.10. If G is a plane graph, then $\sum_{f \in F} d(f) = 2m$.

Proof. By the definition of dual, $d(f) = d(f^*)$ for all $f \in F(G)$ and for corresponding $f^* \in V(G^*)$. So

$$\begin{aligned} \sum_{f \in F(G)} d(f) &= \sum_{f^* \in V(G^*)} d(f^*) = 2e(G^*) \text{ by Theorem 1.1 applied to } G^* \\ &= 2e(G) \text{ since } e(G) = e(G^*) \text{ by equation (10.1)} \\ &= 2m. \end{aligned}$$

□

Proposition 10.11

Proposition 10.11. A simple connected plane graph is a triangulation if and only if its dual is cubic.

Proof. Let G be a simple connected plane graph.

If G is a triangulation then by definition each face of G is degree three. So in the dual G^*, each vertex is of degree three (since $d(f) = d(f^*)$ for each $f \in F(G)$ by equation (10.1)) and hence G^* is cubic.

If G^* is cubic, then for each vertex f^* of G^* we have $d(f^*) = 3$. So in connected plane graph G, $d(f) = 3$ for each face f of G. That is, G is a triangulation. □

Proposition 10.12

Proposition 10.12. Let G be a connected plane graph and let e be an edge of G that is not a cut edge. Then $(G \setminus e)^* \cong G^*/e^*$.

Proof. Because e is not a cut edge then by Note 10.2.A, the two faces f_1 and f_2 of G incident with e are distinct. Deleting e from G results in the "amalgamation" of f_1 and f_2 into a single face f (in $\tilde{G}$; see Figure 10.12).

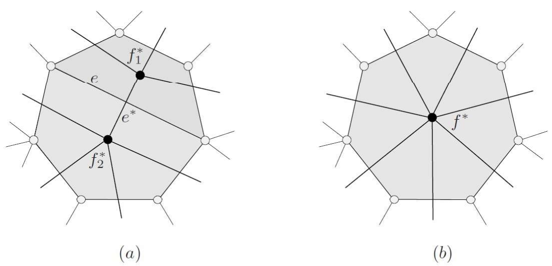


Figure 10.12. (a) G and G^*, (b) $G \setminus e$ and G^*/e^*.

Proposition 10.12 (continued 1)

Proof (continued). Any face of G that is adjacent to f_1 or f_2 in G is adjacent to f in $G \setminus e$. All other faces and adjacencies between faces are unaffected by the deletion of edge e. In the dual plane graph G^* the two vertices f_1^* and f_2^* corresponding to the faces f_1 and f_2 of G are identified (we denote the common identified vertex as f^*) after the deletion of edge e (that is, we create the graph G^*/e^*; see Note 10.2.B for properties of e^*).

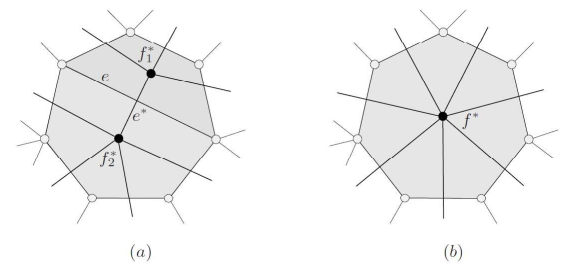


Figure 10.12. (a) G and G^*, (b) $G \setminus e$ and G^*/e^*.

Proposition 10.12 (continued 2)

Proposition 10.12. Let G be a connected plane graph and let e be an edge of G that is not a cut edge. Then $(G \setminus e)^* \cong G^*/e^*$.

Proof (continued). Any vertex of G^* that is adjacent to f_1^* or adjacent to f_2^* is adjacent to f^* in G^*/e^* (as is the case for the corresponding face of $G \setminus e$). Other adjacencies between faces of $G \setminus e$ (and corresponding vertices of $(G \setminus e)^*$) are the same as in G (and corresponding vertices in G^*). So $(G \setminus e)^* \cong G^*/e^*$, as claimed. □

Proposition 10.13

Proposition 10.13. Let G be a connected plane graph and let e be a link (i.e., a nonloop) of G. Then $(G/e)^* \cong G^* \setminus e^*$.

Proof. Since G is connected then by Exercise 10.2.4 (or Exercise 10.2.6 in some printings of the book) we have that $G^{**} \cong G$. Since edge e is not a loop of G by hypothesis, then by Note 10.2.B (actually the contrapositive of Note 10.2.B) the edge e^* is not a cut edge of G^*. So $G^* \setminus e^*$ is connected. By Proposition 10.12, applied to graph G^* and edge e^*,

$$(G^* \setminus e^*)^* \cong G^{**}/e^{**} \cong G/e.$$

Since $G^{**} \cong G$, then taking duals we have

$$G^* \setminus e^* \cong ((G^* \setminus e^*)^*)^* \cong (G/e)^*,$$

as claimed. □

Theorem 10.14

Theorem 10.14. The dual of a nonseparable plane graph is nonseparable.

Proof. Let G be a nonseparable plane graph. If G has no edges then G^* is the trivial graph K_1 and K_1 is nonseparable, so the result holds for graphs with 0 edges. If G has one edge then G is isomorphic to K_1 with a loop attached (which has dual $G^* \cong K_2$ and K_2 is nonseparable) or G is isomorphic to K_2 (which has dual K_1 with a loop attached and K_1 with a loop is nonseparable). We now give an inductive proof on the number of edges of G. We have the base cases of 0 edges and 1 edge established. So let G be a nonseparable graph with $m \geq 2$ edges and suppose all nonseparable plane graphs with less than m edges have nonseparable duals. Then by Note 5.2.A, G is loopless. By Theorem 5.2, any two edges of a nonseparable graph lie in a common cycle, so G has no cut edges. Let e be an edge of G. Then by Exercise 5.3.2, either $G \setminus e$ or G/e is nonseparable.

Theorem 10.14 (continued)

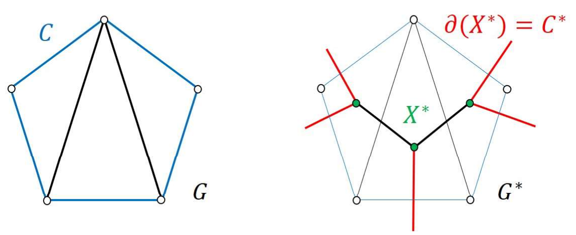
Theorem 10.14. The dual of a nonseparable plane graph is nonseparable.

Proof (continued). First, if $G \setminus e$ is nonseparable (of course $G \setminus e$ is a plane graph since G is) then by Proposition 10.12, $(G \setminus e)^* \cong G^*/e^*$ and so by the induction hypothesis G^*/e^* (as the dual of $G \setminus e$) is nonseparable. As observed above, e^* is then neither a loop nor a cut edge of G^*. Then by Exercise 5.2.2(b), we have that G^* is nonseparable. Second, if G/e is nonseparable (by Exercise 10.1.4(b), G/e is a plane graph since G is) then by Proposition 10.13, $(G/e)^* \cong G^*/e^*$ and so by the induction hypothesis G^*/e^* (as a dual of G/e) is nonseparable. As observed above, e^* is then neither a loop nor a cut edge of G^*. Then by Exercise 5.2.2(a), we have that G^* is separable. In either case (that is, $G \setminus e$ or G/e is nonseparable) we have G^* is nonseparable. Therefore, by induction the result holds for all $m \in \mathbb{N}$ (the number of edges in nonseparable plane graph G), as claimed. □

Theorem 10.16

Theorem 10.16. Let G be a connected plane graph, and let G^* be a plane dual of G. (a) If C is a cycle of G, then C^* is a bond of G^*.

Proof. Let C be a cycle of G, and let X^* denote the set of vertices of G^* that lie in $\text{int}(C)$ (so these vertices correspond to the faces of G in $\text{int}(C)$). Then the edge cut $\partial(X^*)$ in G^* satisfies $\partial(X^*) = C^*$.



By Proposition 10.15, the subgraph of G^* induced by X^*, $G^*[X^*]$, is connected. By Note 10.2.D, the subgraph of G^* induced by $V(G^*) \setminus X^*$ (the vertices in $\text{ext}(C)$) is also connected. Therefore, by Theorem 2.15, C^* is a bond of G^*. □