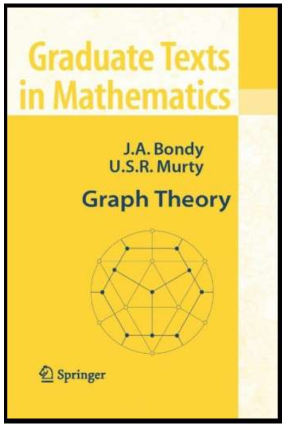
# Graph Theory

**Chapter 10. Planar Graphs**

10.3. Euler's Formula—Proofs of Theorems








## Theorem 10.19

**Theorem 10.19.** Euler's Formula.
For a connected plane graph $G$, $v(G) - e(G) + f(G) = 2$.

**Proof.** If $f(G) = 1$ then each edge of $G$ is a cut edge by Note 10.2.A. Therefore $G$ can contain no cycles; that is, $G$ is a connected acyclic graph, so $G$ is a tree in this case. By Theorem 4.3, this implies $e(G) = v(G) - 1$. Hence, $v(G) - e(G) = 1$ and, since $f(G) = 1$, $v(G) - e(G) + f(G) = 2$. So this claim holds for all graphs with one face. We now give a proof based on mathematical induction and the number of faces $f$ of a graph. We have established the base case of $f = 1$. Now suppose Euler's Formula holds for all connected plane graphs with fewer than $f$ faces where $f \geq 2$. Let $G$ be a connected plane graph with $f$ faces where $f \geq 2$.





## Theorem 10.19 (continued)

**Theorem 10.19.** Euler's Formula.
For a connected plane graph $G$, $v(G) - e(G) + f(G) = 2$.

**Proof (continued).** Choose an edge $e$ of $G$ that is not a cut edge of $G$ (if all edges are cut edges then, as explained above, $G$ is a tree and $f = 1$; since $f \geq 2$ then such an edge exists). Then $G \setminus e$ is a connected plane graph with $f - 1$ faces (since the two faces incident to $e$ in $G$ are coalesced in $G \setminus e$). By the induction hypothesis, $v(G \setminus e) - e(G \setminus e) + f(G \setminus e) = 2$. Since $v(G \setminus e) = v(G)$, $e(G \setminus e) = e(G) - 1$, and $f(G \setminus e) = f(G) - 1$, then $v(G) - (e(G) - 1) + (f(G) - 1) = 2$ or $v(G) - e(G) + f(G) = 2$. So, by induction, Euler's Formula holds for all connected plane graphs. □





## Corollary 10.20

**Corollary 10.20.** All planar embeddings of a connected planar graph have the same number of faces.

**Proof.** Let $\tilde{G}$ be a planar embedding of a planar graph $G$. By Euler's Formula (Theorem 10.19) we have $f(\tilde{G}) = e(\tilde{G}) - v(\tilde{G}) + 2$, and since $e(\tilde{G}) = e(G)$ and $v(\tilde{G}) = v(G)$ then $f(\tilde{G}) = e(G) - v(G) + 2 = f(G)$. So for any planar embedding $\tilde{G}$ of $G$, we have $f(\tilde{G}) = f(G)$, as claimed. □

## Corollary 10.21

**Corollary 10.21.** Let $G$ be a simple planar graph on at least three vertices. Then $m \leq 3n - 6$. Furthermore, $m = 3n - 6$ if and only if every planar embedding of $G$ is a triangulation.

**Proof.** It suffices to prove the result for connected graphs (since for a graph with $k$ components, we can introduce $m_1, m_2, \ldots, m_k$ and $n_1, n_2, \ldots, n_k$ for the numbers of edges and vertices in the components, and then apply the result to each component). Let $G$ be a simple connected planar graph with $n \geq 3$. Let $\tilde{G}$ be any planar embedding of $G$. Since $G$ is simple and connected with at least 3 vertices, then $\tilde{G}$ has no faces of degree 2; that is, $d(f) \geq 3$ for all $f \in F(\tilde{G})$. Therefore

$$\begin{aligned} 2m &= \sum_{f \in F(\tilde{G})} d(f) \text{ by Theorem 10.10} \\ &\geq 3f(\tilde{G}) \text{ since } f(\tilde{G}) \text{ is the number of faces in } \tilde{G} \\ &= 3(m - n + 2) \text{ by Euler's Formula (Theorem 10.19).} \end{aligned}$$





## Corollary 10.21 (continued)

**Corollary 10.21.** Let $G$ be a simple planar graph on at least three vertices. Then $m \leq 3n - 6$. Furthermore, $m = 3n - 6$ if and only if every planar embedding of $G$ is a triangulation.

**Proof (continued).** So $m \leq 3n - 6$, as claimed.

Now equality holds if and only if $\sum_{f \in F(\tilde{G})} d(f) = 3f(\tilde{G})$; that is, if and only if $d(f) = 3$ for each $f \in F(\tilde{G})$ (since $d(f) \geq 3$, as shown above). That is, equality holds if and only if $\tilde{G}$ is (by definition) a triangulation, as claimed. □





## Corollary 10.22

**Corollary 10.22.** Every simple planar graph has a vertex of degree at most five.

**Proof.** Since a simple planar graph on $n < 3$ has at most 2 edges, the result holds for $n < 3$. If $n \geq 3$ then

$$\begin{aligned} \delta n &\leq \sum_{v \in V} d(v) \\ &= 2m \text{ by Theorem 1.1} \\ &\leq 6n - 12 \text{ by Corollary 10.21.} \end{aligned}$$

So $\delta \leq 6 - 12/n < 6$ and so $\delta \leq 5$ (since $\delta \in \mathbb{N}$), as claimed. □





## Corollary 10.23

**Corollary 10.23.** $K_5$ are nonplanar.

**Proof.** ASSUME $K_5$ is planar. Since $K_5$ is simple and connected, then Corollary 10.21 implies $10 = e(K_5) \leq 3v(K_5) - 6 = 9$, a CONTRADICTION. So $K_5$ is nonplanar. □

# Corollary 10.24

**Corollary 10.24.** $K_{3,3}$ is nonplanar.

**Proof.** ASSUME that $K_{3,3}$ is planar and let $\tilde{G}$ be a planar embedding of $K_{3,3}$. Since $K_{3,3}$ is simple then it has no cycles of length 2 and since $K_{3,3}$ is bipartite then it has no cycles of length 3 (by Theorem 4.7). That is $K_{3,3}$ has no cycle of length less than four, so that every face of $G$ has degree at least four. Then

$$\begin{aligned} 4f(\tilde{G}) &\leq \sum_{f \in F(\tilde{G})} d(f) \\ &= 2e(\tilde{G}) \text{ by Theorem 10.10} \\ &= 18. \end{aligned}$$

But this implies $f(\tilde{G}) \leq 9/2$, or $f(\tilde{G}) \leq 4$ since $f(\tilde{G}) \in \mathbb{N}$. Then be Euler's Formula (Theorem 10.19), $2 = v(\tilde{G}) - e(\tilde{G}) + f(\tilde{G}) \leq 6 - 9 + 4 = 1$, a CONTRADICTION. So $K_{3,3}$ is nonplanar. □