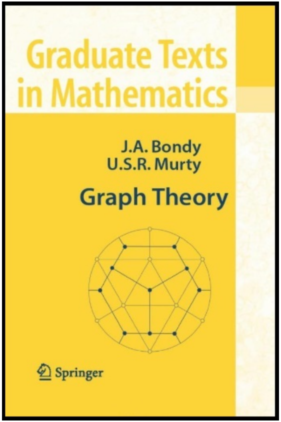
Graph Theory

Chapter 10. Planar Graphs

10.3. Euler's Formula—Proofs of Theorems



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Proof. If $f(G) = 1$ then each edge of G is a cut edge by Note 10.2.A. Therefore G can contain no cycles; that is, G is a connected acyclic graph, so G is a tree in this case. By Theorem 4.3, this implies $e(G) = v(G) - 1$. Hence, $v(G) - e(G) = 1$ and, since $f(G) = 1$, $v(G) - e(G) + f(G) = 2$. So this claim holds for all graphs with one face.

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Theorem 10.19 (continued)

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For a connected plane graph G, $v(G) - e(G) + f(G) = 2$.

Proof (continued). Choose an edge e of G that is not a cut edge of G (if all edges are cut edges then, as explained above, G is a tree and $f = 1$; since $f \geq 2$ then such an edge exists). Then $G \setminus e$ is a connected plane graph with $f - 1$ faces (since the two faces incident to e in G are coalesced in $G \setminus e$).

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Corollary 10.20

Corollary 10.20. All planar embeddings of a connected planar graph have the same number of faces.

Proof. Let $\tilde{G}$ be a planar embedding of a planar graph G. By Euler's Formula (Theorem 10.19) we have $f(\tilde{G}) = e(\tilde{G}) - v(\tilde{G}) + 2$, and since $e(\tilde{G}) = e(G)$ and $v(\tilde{G}) = v(G)$ then $f(\tilde{G}) = e(G) - v(G) + 2 = f(G)$. So for any planar embedding $\tilde{G}$ of G, we have $f(\tilde{G}) = f(G)$, as claimed. □

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Corollary 10.21

Corollary 10.21. Let G be a simple planar graph on at least three vertices. Then $m \leq 3n - 6$. Furthermore, $m = 3n - 6$ if and only if every planar embedding of G is a triangulation.

Proof. It suffices to prove the result for connected graphs (since for a graph with k components, we can introduce $m_1, m_2, \ldots, m_k$ and $n_1, n_2, \ldots, n_k$ for the numbers of edges and vertices in the components, and then apply the result to each component). Let G be a simple connected planar graph with $n \geq 3$.

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$$\begin{aligned} 2m &= \sum_{f \in F(\tilde{G})} d(f) \text{ by Theorem 10.10} \\ &\geq 3f(\tilde{G}) \text{ since } f(\tilde{G}) \text{ is the number of faces in } \tilde{G} \\ &= 3(m - n + 2) \text{ by Euler's Formula (Theorem 10.19).} \end{aligned}$$

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Now equality holds if and only if $\sum_{f \in F(\tilde{G})} d(f) = 3f(\tilde{G})$; that is, if and only if $d(f) = 3$ for each $f \in F(\tilde{G})$ (since $d(f) \geq 3$, as shown above). That is, equality holds if and only if $\tilde{G}$ is (by definition) a triangulation, as claimed. □

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Corollary 10.22

Corollary 10.22. Every simple planar graph has a vertex of degree at most five.

Proof. Since a simple planar graph on $n < 3$ has at most 2 edges, the result holds for $n < 3$. If $n \geq 3$ then

$$\begin{aligned} \delta n &\leq \sum_{v \in V} d(v) \\ &= 2m \text{ by Theorem 1.1} \\ &\leq 6n - 12 \text{ by Corollary 10.21.} \end{aligned}$$

So $\delta \leq 6 - 12/n < 6$ and so $\delta \leq 5$ (since $\delta \in \mathbb{N}$), as claimed. □

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Corollary 10.23. K_5 are nonplanar.

Proof. ASSUME K_5 is planar. Since K_5 is simple and connected, then Corollary 10.21 implies $10 = e(K_5) \leq 3v(K_5) - 6 = 9$, a CONTRADICTION. So K_5 is nonplanar. □

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Corollary 10.24

Corollary 10.24. $K_{3,3}$ is nonplanar.

Proof. ASSUME that $K_{3,3}$ is planar and let $\tilde{G}$ be a planar embedding of $K_{3,3}$. Since $K_{3,3}$ is simple then it has no cycles of length 2 and since $K_{3,3}$ is bipartite then it has no cycles of length 3 (by Theorem 4.7). That is $K_{3,3}$ has no cycle of length less than four, so that every face of G has degree at least four.

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$$\begin{aligned} 4f(\tilde{G}) &\leq \sum_{f \in F(\tilde{G})} d(f) \\ &= 2e(\tilde{G}) \text{ by Theorem 10.10} \\ &= 18. \end{aligned}$$

But this implies $f(\tilde{G}) \leq 9/2$, or $f(\tilde{G}) \leq 4$ since $f(\tilde{G}) \in \mathbb{N}$.

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