Graph Theory

Chapter 10. Planar Graphs

10.4. Bridges—Proofs of Theorems

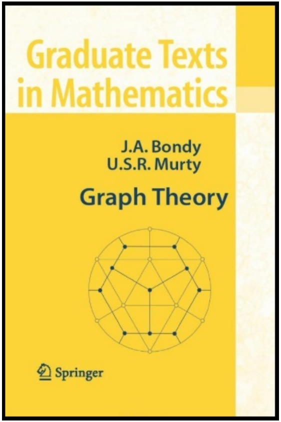

Table of contents

1. Theorem 10.25
2. Theorem 10.26
3. Theorem 10.27
4. Theorem 10.28

Theorem 10.25

Theorem 10.25. Overlapping bridges of a cycle in a connected graph are either skew or else equivalent 3-bridges.

Proof. Suppose that bridges B and B' of the cycle overlap. Then, by definition, both B and B' must have at least two vertices of attachment. If either B or B' is a 2-bridge then, since the bridges do not avoid each other (so that all vertices of attachment of one does not lie in a single segment of the other, so the two segments of C determined by a 2-bridge must each contain some vertex of attachment of the other bridge) then they are skew.

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Theorem 10.25 (continued)

Proof (continued). If B and B' are not equivalent (that is, they do not have exactly the same vertices of attachment), then B' has a vertex u' of attachment strictly between two consecutive vertices of attachment u and v of B. Because B and B' do not avoid each other (that is, they overlap) then some vertex of attachment v' of B' does not lie in the segment of B connecting u and v (and so $v' \neq u$ and $v' \neq v$). Therefore B and B' are skew. That is, if B and B' are not equivalent then they are skew.

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If B and B' are equivalent k-bridges (that is, they have the same k vertices of attachment) then $k \geq 3$ (if $k = 2$, then the segments determined by B and B' are the same and then B and B' avoid each other, contradicting the hypothesis that B and B' overlap). If $k \geq 4$, then B and B' are skew (take 4 vertices of attachment in the order determined by cycle C, associate the first and third ones with B and the second and forth one with B').

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Theorem 10.26

Theorem 10.26. Let G be a plane graph containing cycle C. The inner bridge of C avoid one another, and the outer bridges of C avoid one another.

Proof. Let B and B' be inner bridges of cycle C. ASSUME B and B' overlap. By Theorem 10.25, B and B' are either skew or equivalent 3-bridges.

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Case 1. Suppose B and B' are skew. Then there are, by definition of "skew," distinct vertices u, v in B and u', v' in B' appearing in the cyclic order u, u', v, v' on C. Let uPv be a path in B and $u'P'v'$ a path in B', both internally disjoint from C (the paths exist since bridges by definition are either single edges or connected components F of $G - V(C)$). Consider the subgraph $H = C \cup P \cup P'$ of G (see Figure 10.17, left).

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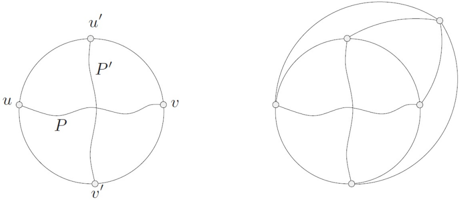

Because G is a plane graph then so is a subgraph H. Let K be the plane graph obtained from H by adding a vertex in ext(C) and joining it to u, u', v, v' (with no crossings; see Figure 10.17, right). Then K is a subdivision of K_5 (a “subdivision since P and P' may not be paths of length 1 and the paths in C determined by u, u', v, v' may not be of length 1). But K_5 is nonplanar by Corollary 10.23 and this implies that H is a nonplanar subgraph of G (this is spelled out in Kuratowski's Theorem, Theorem 10.30, in the next section), a CONTRADICTION to the fact that G is a plane graph.

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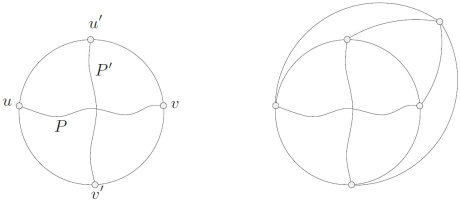

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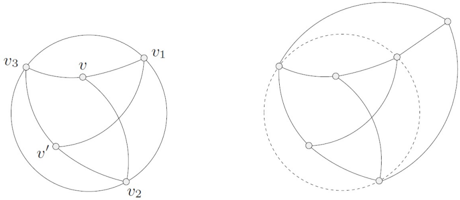
Theorem 10.26 (continued 2)

Proof (continued).
Case 2. Suppose B and B' are equivalent 3-bridges. Let the common set of vertices of attachment be $S = \{v_1, v_2, v_3\}$. By Exercise 9.2.3, there is a (v, S)-fan F in B for some internal vertex v of B. Similarly, there is a (v', S)-fan F' in B' for some internal vertex v' of B'. Consider the subgraph $H = F \cup F'$ of G (see Figure 10.18, left).



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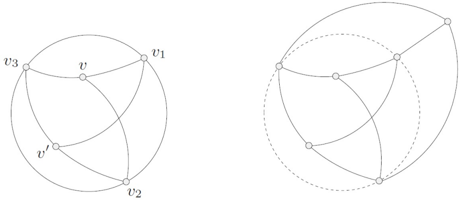

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Proof (continued). This implies that H is a nonplanar subgraph of G, a CONTRADICTION to the fact that G is a plane graph.

Since we get a contradiction in both cases, then the assumption that B and B' overlap is false and hence B and B' avoid one another. The proof for outer bridges is similar. □

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Theorem 10.27. A cycle in a simple 3-connected plane graph is a facial cycle if and only if it is nonseparating.

Proof. Let G be a simple 3-connected plane graph and let C be a cycle of G. Suppose that C is not a facial cycle of G. Then C has at least one inner bridge (or else C would be the boundary of the face $\text{int}(C)$) and at least one outer bridge (or else C would be the boundary of the face of $\text{ext}(C)$). Since G is simple and connected, these bridges are not loops. So either they are both nontrivial (in which case C is not a nonseparating cycle) or at least one of them is a chord (in which case C is not a facial cycle then it is not nonseparating (or equivalently, if C is nonseparating cycle then C is a facial cycle).

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Now suppose C is a facial cycle of G. By Proposition 10.5, we may assume without loss of generality that C bounds the outer face of G. So all bridges of C are inner bridges. By Theorem 10.26, these bridges avoid one another.

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Proof (continued). ASSUME C has a chord xy (which would be a trivial bridge; notice that x and y are not adjacent in C since G is simple). Then there are two xy-segments of C and each has at least one internal vertex. Since the bridges avoid one another, there can be no bridge joining an internal vertex of one xy-segment to an internal vertex of the other xy-segment. Therefore $\{x, y\}$ is a vertex cut which separates the internal vertices of the two xy-segments. But G is 3-connected and so cannot have a vertex cut of size 2, a CONTRADICTION. So C has no chords.

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Theorem 10.28. Every simple 3-connected planar graph has a unique planar embedding.

Proof. Let G be a simple 3-connected planar graph. By Theorem 10.27, the facial cycles in any planar embedding of G are precisely its nonseparating cycles. Now a separating cycle is a cycle with no chords and at most one nontrivial bridge. The existence of a chord of a cycle and the definition of nontrivial bridge are independent of any planar embedding of G (they are part "of the abstract structure of the graph," as Bondy and Murty say).

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