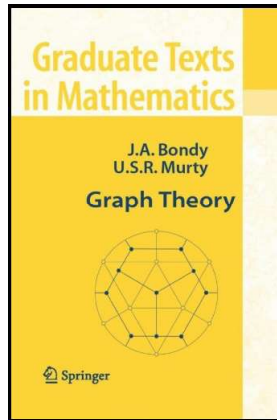


Graph Theory

Chapter 10. Planar Graphs

10.5. Kuratowski's Theorem—Proofs of Theorems



Lemma 10.33

Lemma 10.33. Let G be a graph with a 2-vertex cut set $S = \{x, y\}$. Then each marked $\{x, y\}$ -component of G is isomorphic to a minor of G .

Proof. Let H be an S -component of G , with marker edge e , and let xPy be a path in another S -component of G (which exists since the S -components are connected). Then $H \cup P$ is a subgraph of G . Also, $H \cup P$ is isomorphic to a subdivision of $G + e$ (just subdivide edge $e = xy$ until it is isomorphic to path P joining x and y). So $G + e$ is isomorphic to a minor of G . \square

Lemma 10.34

Lemma 10.34. Let G be a graph with a 2-vertex cut set $S = \{x, y\}$. Then G is planar if and only if each of its marked S -components is planar.

Proof. Suppose G is planar. By Lemma 10.33, each marked S -component of G is isomorphic to a minor of G . Then by Proposition 10.31, the minor is planar and so the marked S -component is planar.

Now suppose that G has marked S -components each of which are planar. Let $e = xy$ denote their common marker edge. Exercise 10.4.1 states that if G_1 and G_2 are planar with intersection isomorphic to K_2 , then $G_1 \cup G_2$ is planar. So by this (and induction) the union of the marked S -components of G is planar. This union is $G + e$, so $G + e$ is planar. Therefore, by Note 10.2.C, G is planar. \square

Theorem 10.35

Theorem 10.35. Every 3-connected nonplanar graph has a Kuratowski minor.

Proof. Let G be a 3-connected nonplanar graph. Since we are making a claim about a minor graph, then we may assume that G is simple (for G can be made simple by edge deletion and if the result holds for simple graphs then it holds for nonsimple graphs). All graphs on four or fewer vertices are planar, so without loss of generality we can assume $n \geq 5$. We give an inductive proof on n , with the result holding trivially for $n \leq 4$ (the base case). Suppose the result holds for all 3-connected nonplanar graphs on n vertices or less and let G be a 3-connected nonplanar graph with $n + 1$ vertices. By Theorem 9.10, G contains an edge $e = xy$ such that $H = G/e$ is 3-connected. By the induction hypothesis, if H is nonplanar then it has a Kuratowski minor. Since every minor of G/e is also a minor of G , we deduce that G too has a Kuratowski minor and we are done. So we can assume, without loss of generality, that H is planar.

Theorem 10.35 (continued 1)

Theorem 10.35. Every 3-connected nonplanar graph has a Kuratowski minor.

Proof (continued). Let \tilde{H} be a plane embedding of H . Denote by z the vertex of H formed by contracting edge e described above. Because H is simple then it is loopless and 3-connected, so by Corollary 10.8 the neighbors of z lie on a cycle C . Then C is the boundary of a face f of $\tilde{H} - z$ where f as a subset of \mathbb{R}^2 containing vertex z in the plane embedding \tilde{H} of H (as shown in the proof of Corollary 10.8 — see the last line of the proof). Now consider the edge deleted graph $G \setminus e$ which also contains cycle C . Now x and y are vertices in $G \setminus e$ which are not in cycle C , so there are bridges of C in $G \setminus e$, B_x and B_y , that contain x and y , respectively. Notice that all neighbors of x and all neighbors of y lie on cycle C . Also, cycle C is the boundary of some face f of $\tilde{H} - z$, so face f cannot contain any other edges of $\tilde{H} - z$ other than those that join C to x or join C to y .

Theorem 10.35 (continued 2)

Proof (continued). ASSUME first that B_x and B_y avoid each other (so the vertices of attachment of one bridge lie in a single segment of the cycle determined by the other bridge). Then (here comes a soft argument) B_x and B_y can be embedded in face f of $\tilde{H} - z$ in such a way that vertices x and y belong to the same face of the resulting graph $(\tilde{H} - z) \cup \tilde{B}_x \cup \tilde{B}_y$ (see Figure 10.20).

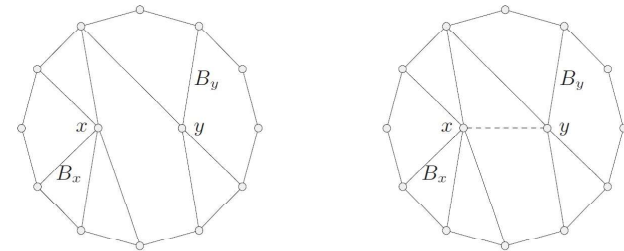


Figure 10.20. A planar embedding of $G \setminus e$ (left) and G (right) where B_x and B_y avoid each other.

Theorem 10.35 (continued 3)

Proof (continued). Informally, we have \tilde{B}_x and \tilde{B}_y are stars with centers x and y , respectively, and their edges have no crossings (though they may share one or two vertices in their coronas). The edge xy can now be added to $(\tilde{H} - z) \cup \tilde{B}_x \cup \tilde{B}_y$ without any crossings to give a planar embedding of graph G , a CONTRADICTION to the hypothesis that G is nonplanar. Hence, bridges B_x and B_y do not avoid each other (i.e., they overlap).

With bridges B_x and B_y overlapping, by Theorem 10.25 they are either skew (in which case some of their vertices “alternate” on cycle C) or they are equivalent 3-bridges (in which case they have the same three vertices of attachment to C). In the first case, G has a $K_{3,3}$ minor (which includes edge xy) and in the second case G has a K_5 minor (which includes edge xy). See Figure 10.21.

Theorem 10.35 (continued 4)

Proof (continued).

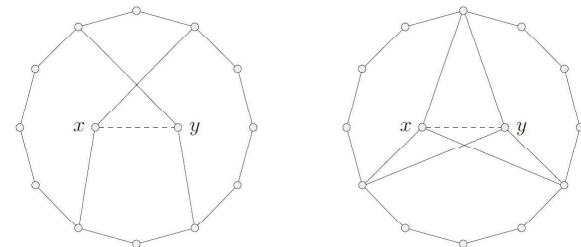


Figure 10.21. G has a $K_{3,3}$ -minor when B_x and B_y are skew (left), and G has a K_5 -minor when B_x and B_y are equivalent 3-bridges (right).

Therefore the result holds for graph G on $n + 1$ vertices, and by mathematical induction the result holds for all 3-connected nonplanar graphs, as claimed. \square

Theorem 10.32

Theorem 10.32. Wagner's Theorem. A graph is planar if and only if it has no Kuratowski minor.

Proof. Let G be a graph (without loss of generality, G is connected and so is 1-connected).

First, suppose G is planar. By Proposition 10.31, a planar graph cannot have a nonplanar minor so G does not have a Kuratowski minor.

Second, suppose G is nonplanar. If G is k -connected for $k \geq 3$, then G is also 3-connected and so G has a Kuratowski minor by Theorem 10.35. If G is not 3-connected then G has a 2-vertex cut $\{x, y\}$ (notice that this holds if G is either 1-connected or 2-connected. . . just as long as it is *not* 3-connected). Now the decomposition tree of G consists of leaves which are either 3-connected graphs or graphs whose underlying simple graph is K_3 (see the definition of "decomposition tree" and Figure 9.8 in [Section 9.4. Three Connected Graphs](#)).

Theorem 10.32 (continued)

Theorem 10.32. Wagner's Theorem. A graph is planar if and only if it has no Kuratowski minor.

Proof (continued). By Lemma 10.34, since G is nonplanar then it has a marked $\{x, y\}$ -component (see Section 9.4) that is not planar. By Lemma 10.33, each marked $\{x, y\}$ -component of G is isomorphic to a minor of G , so G has a nonplanar minor. Since this minor cannot have K_3 as its underlying simple graph, then it must be a 3-connected nonplanar graph. So by Theorem 10.35, this minor has a Kuratowski minor and so G itself has a Kuratowski minor, as claimed. \square