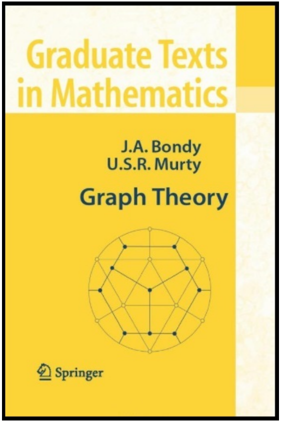
Graph Theory

Chapter 10. Planar Graphs

10.5. Kuratowski's Theorem—Proofs of Theorems



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Now suppose that G has marked S-components each of which are planar. Let $e = xy$ denote their common marker edge. Exercise 10.4.1 states that if G_1 and G_2 are planar with intersection isomorphic to K_2, then $G_1 \cup G_2$ is planar. So by this (and induction) the union of the marked S-components of G is planar. This union is $G + e$, so $G + e$ is planar. Therefore, by Note 10.2.C, G is planar. □

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Theorem 10.35

Theorem 10.35. Every 3-connected nonplanar graph has a Kuratowski minor.

Proof. Let G be a 3-connected nonplanar graph. Since we are making a claim about a minor graph, then we may assume that G is simple (for G can be made simple by edge deletion and if the result holds for simple graphs then it holds for nonsimple graphs). All graphs on four or fewer vertices are planar, so without loss of generality we can assume $n \geq 5$. We give an inductive proof on n, with the result holding trivially for $n \leq 4$ (the base case). Suppose the result holds for all 3-connected nonplanar graphs on n vertices or less and let G be a 3-connected nonplanar graph with $n+1$ vertices.

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Proof (continued). Let $\tilde{H}$ be a plane embedding of H. Denote by z the vertex of H formed by contracting edge e described above. Because H is simple then it is loopless and 3-connected, so by Corollary 10.8 the neighbors of z lie on a cycle C. Then C is the boundary of a face f of $\tilde{H} - z$ where f as a subset of $\mathbb{R}^2$ containing vertex z in the plane embedding $\tilde{H}$ of H (as shown in the proof of Corollary 10.8 — see the last line of the proof). Now consider the edge deleted graph $G \setminus e$ which also contains cycle C. Now x and y are vertices in $G \setminus e$ which are not in cycle C, so there are bridges of C in $G \setminus e$, B_x and B_y, that contain x and y, respectively. Notice that all neighbors of x and all neighbors of y lie on cycle C. Also, cycle C is the boundary of some face f of $\tilde{H} - z$, so face f cannot contain any other edges of $\tilde{H} - z$ other than those that join C to x or join C to y.

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Proof (continued). ASSUME first that B_x and B_y avoid each other (so the vertices of attachment of one bridge lie in a single segment of the cycle determined by the other bridge). Then (here comes a soft argument) B_x and B_y can be embedded in face f of $\tilde{H} - z$ in such a way that vertices x and y belong to the same face of the resulting graph $(\tilde{H} - z) \cup \tilde{B}_x \cup \tilde{B}_y$ (see Figure 10.20).

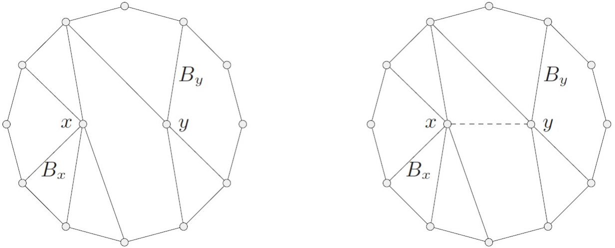

Figure 10.20. A planar embedding of $G \setminus e$ (left) and G (right) where B_x and B_y avoid each other.

Theorem 10.35 (continued 3)

Proof (continued). Informally, we have $\tilde{B}_x$ and $\tilde{B}_y$ are stars with centers x and y, respectively, and their edges have no crossings (though they may share one or two vertices in their coronas). The edge xy can now be added to $(\tilde{H} - z) \cup \tilde{B}_x \cup \tilde{B}_y$ without any crossings to give a planar embedding of graph G, a CONTRADICTION to the hypothesis that G is nonplanar. Hence, bridges B_x and B_y do not avoid each other (i.e., they overlap).

With bridges B_x and B_y overlapping, by Theorem 10.25 they are either skew (in which case some of their vertices "alternate" on cycle C) or they are equivalent 3-bridges (in which case they have the same three vertices of attachement to C). In the first case, G has a $K_{3,3}$ minor (which includes edge xy) and in the second case G has a K_5 minor (which includes edge xy). See Figure 10.21.

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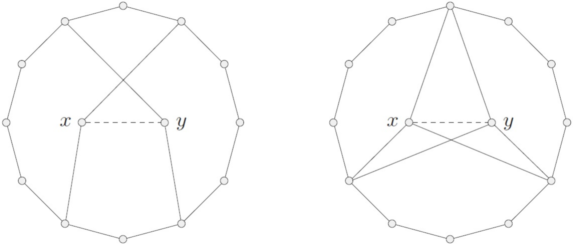

Figure 10.21. G has a $K_{3,3}$-minor when B_x and B_y are skew (left), and G has a K_5-minor when B_x and B_y are equivalent 3-bridges (right).

Therefore the result holds for graph G on $n+1$ vertices, and by mathematical induction the result holds for all 3-connected nonplanar graphs, as claimed. □

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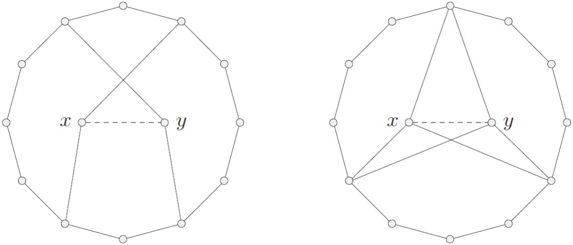

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