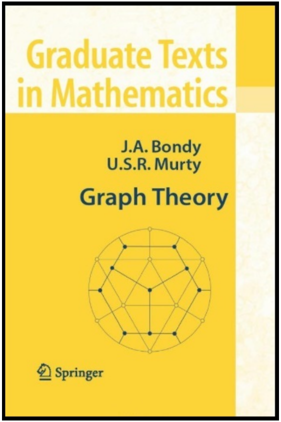
Graph Theory

Chapter 11. The Four-Colour Problem

11.1. Colourings of Planar Maps—Proofs of Theorems



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First suppose that G has a proper 4-face-colouring. Denote the colours by the vectors $\alpha_0 = (0,0)$, $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$, and $\alpha_3 = (1,1)$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$. We'll give a 3-edge-colouring of G by assigning to each edge the sum of the colours of the two faces it separates. Notice that since G has no cut edges then each edge separates two distinct faces, so no edge is assigned colour α_0 under this scheme (since each vector is its own additive inverse in $\mathbb{Z}_2 \times \mathbb{Z}_2$).

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Theorem 11.4 (continued 1)

Proof (continued).

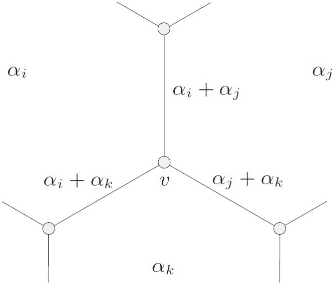

Figure 11.2. The 3-edge-colouring of a cubic plane graph induced by a 4-face-colouring.

Since the face colouring is proper then α_i, α_j, and α_k are distinct and hence the three edges incident to v have different colours. Following this scheme to assign colours to all edges of G gives a proper 3-edge-colouring in colours α_1, α_2, α_3, as needed.

Theorem 11.4 (continued 2)

Theorem 11.4. Tait's Theorem.
A 3-connected cubic plane graph is 4-face-colourable if and only if it is 3-edge-colourable.

Proof (continued). Second, suppose that G has a proper 3-edge-colouring in colours 1, 2, 3. Denote by E_i the set of edges of G of colour i, for $i \in \{1, 2, 3\}$. The induced subgraph $G[E_i]$ is a spanning 1-regular subgraph of G (since G is cubic graph with a proper 3-edge-colouring). Set $G_{ij} = G[E_i \cup E_j]$ for $1 \leq i < j \leq 3$. Then each G_{ij} is a spanning 2-regular subgraph of G (since the $G[E_i]$ are spanning with $G[E_i]$ and $G[E_j]$ edge disjoint). Then by Exercise 11.1.2/11.2.2, G_{ij} is 2-face-colourable. Also, each face of G is the intersection of a face of G_{12} and a face of G_{23} (since G_{12} and G_{23} together include all edges; see Figure 11.3 for an illustration of this for the cube Q_2).

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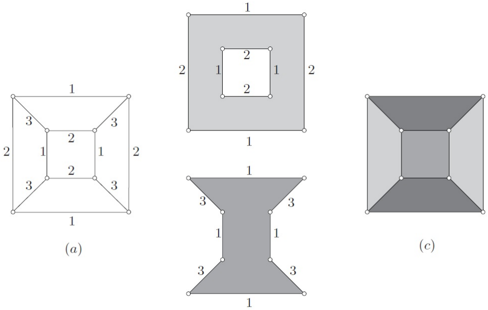

Figure 11.3. (a) A 3-edge-colouring of the cube, (b) 2-face-colourings (shaded and unshaded) of the spanning subgraphs G_{12} and G_{13}, (c) the induced 4-face colouring of the cube.

Consider the 2-face-colourings of G_{12} and G_{23}, say with the face colours 0 and 1 (with colour 0 represented by unshaded faces and colour 1 represented by shaded faces in Figure 11.3(b)).

Theorem 11.4 (continued 4)

Proof (continued). Now assign to each face f of G the ordered pair of colours assigned to faces G_{12} and G_{23} (respectively) whose intersection is f. This gives a 4-face-colouring of G in the colours α_0, α_1, α_2, α_3 mentioned above. Because $G = G_{12} \cup G_{23}$, then this is a proper 4-face-colouring of G (see Figure 11.3(c)), as needed. □

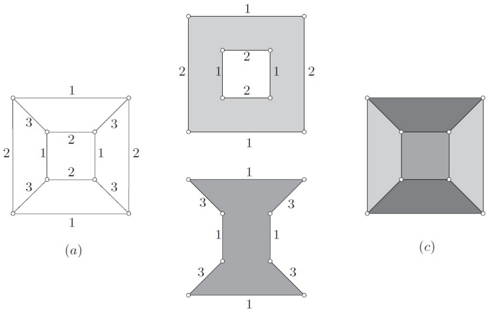

Figure 11.3.