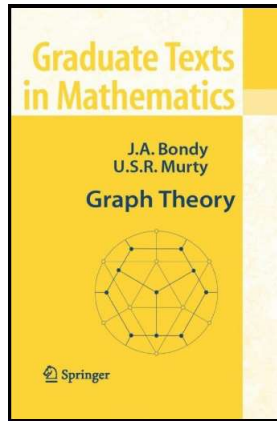


## Graph Theory

## Chapter 11. The Four-Colour Problem

## 11.2. The Five-Colour Theorem—Proofs of Theorems



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## Theorem 11.6

**Theorem 11.6.** THE FIVE-COLOUR THEOREM.

Every loopless planar graph is 5-colourable.

**Proof.** We give a proof based on induction on the number of vertices. As observed in Note 11.1.A, it suffices to prove the result for 3-connected triangulations. So let  $G$  be such a triangulation. The result holds for  $G = K_5 \setminus e$ , so we take this as a base case. For the induction hypothesis, suppose the result holds for all appropriate graphs on  $n - 1$  vertices and let  $G$  have  $n$  vertices. Since  $G$  is simple and planar then by Corollary 10.22 it has a vertex  $v$  of degree at most five. Consider the plane graph  $H = G - v$ . By the induction hypothesis,  $H$  has a proper 5-colouring. If in this colouring of  $H$ , one of the five colourings is assigned to no neighbor of  $v$  then we may assign that colour to  $v$  giving a proper 5-colouring of  $G$  and the result holds for all graphs by induction. So we only need to consider the case where the five neighbors of  $v$  are all different colours. We now show that this is not the case by way of contradiction.

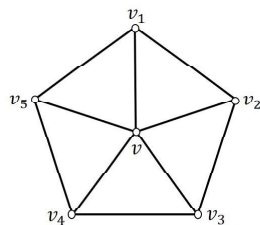
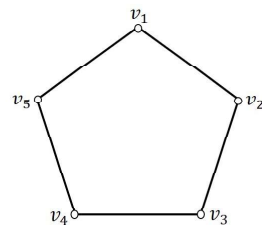
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## Theorem 11.6 (continued 1)

**Proof (continued).** ASSUME that vertices  $v$  of degree five in graph  $G$  has neighbors all of different colours. Let the neighbors of  $v$  be  $v_1, v_2, v_3, v_4, v_5$  and let  $C = v_1 v_2 v_3 v_4 v_5 v_1$  be the facial cycle of  $H$  containing these vertices:

In  $G$ Facial cycle  $C$  in  $H$ 

Let the colours of vertex  $v_i$  be denoted as  $i$  where  $i \in \{1, 2, 3, 4, 5\}$ . We may suppose that the vertex  $v$  of  $G$  lies in  $\text{int}(C)$  (see the image above), so bridges of  $C$  in  $H$  are outer bridges.

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## Theorem 11.6 (continued 2)

**Proof (continued).** If there is no bridge of  $C$  in  $H$  containing both  $v_1$  and  $v_3$ , then by swapping the colours of the vertices coloured 1 and 3 in all the bridges of  $C$  containing  $v_1$  (remember that the bridges are internally disjoint), we obtain a proper 5-colouring of  $H$  in which no vertex of  $C$  has colour 1. This colour may now be assigned to  $v_1$  resulting in a proper 5-colouring of  $G$ . So we may assume without loss of generality that there is a bridge  $B_1$  of  $C$  in  $H$  having  $v_1$  and  $v_3$  as vertices of attachment. Similarly, there is a bridge  $B_2$  of  $C$  in  $H$  having  $v_2$  and  $v_4$  as vertices of attachment. But then the bridges  $B_1$  and  $B_2$  overlap, CONTRADICTING Theorem 10.26 which implies that outer bridges avoid one another. So the assumption that the neighbors of  $v$  are all of a different colour is false. Hence  $v$  can be assigned the “moving colour” as described above, giving a 5-colouring of  $G$ , as claimed.  $\square$

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