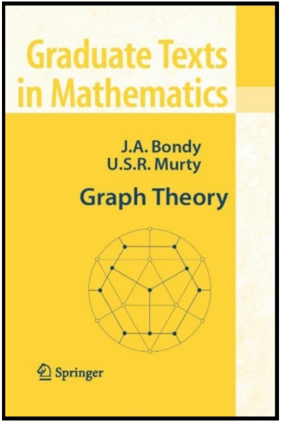
Graph Theory

Chapter 11. The Four-Colour Problem

11.2. The Five-Colour Theorem—Proofs of Theorems



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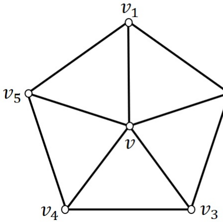
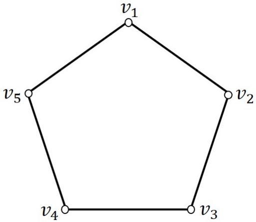
Theorem 11.6 (continued 1)

Proof (continued). ASSUME that vertices v of degree five in graph G has neighbors all of different colours. Let the neighbors of v be v_1, v_2, v_3, v_4, v_5 and let $C = v_1 v_2 v_3 v_4 v_5 v_1$ be the facial cycle of H containing these vertices:



In G



Facial cycle C in H

Let the colours of vertex v_i be denoted as i where $i \in \{1, 2, 3, 4, 5\}$. We may suppose that the vertex v of G lies in int(C) (see the image above), so bridges of C in H are outer bridges.

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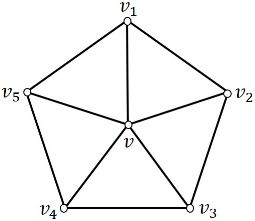

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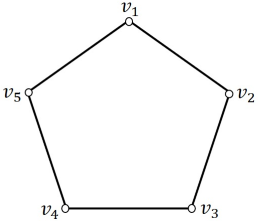

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Theorem 11.6 (continued 2)

Proof (continued). If there is no bridge of C in H containing both v_1 and v_3, then by swapping the colours of the vertices coloured 1 and 3 in all the bridges of C containing v_1 (remember that the bridges are internally disjoint), we obtain a proper 5-colouring of H in which no vertex of C has colour 1. This colour may now be assigned to v_1 resulting in a proper 5-colouring of G. So we may assume without loss of generality that there is a bridge B_1 of C in H having v_1 and v_3 as vertices of attachments. Similarly, there is a bridge B_2 of C in H having v_2 and v_4 as vertices of attachment. But then the bridges B_1 and B_2 overlap, CONTRADICTING Theorem 10.26 which implies that outer bridges avoid one another. So the assumption that the neighbors of v are all of a different colour is false. Hence v can be assigned the "moving colour" as described above, giving a 5-colouring of G, as claimed. □

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