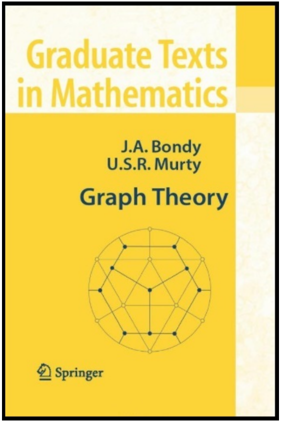
Graph Theory

Chapter 12. Stable Sets and Cliques

12.1. Stable Sets—Proofs of Theorems



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Lemma 12.4. Let $\mathcal{P}$ be a path partition of a digraph D. Suppose that no stable set of D is orthogonal to $\mathcal{P}$. Then there is a path partition $\mathcal{Q}$ of D such that $|\mathcal{Q}| = |\mathcal{P}| - 1$, $i(\mathcal{Q}) \subset i(\mathcal{P})$, and $t(\mathcal{Q}) \subset t(\mathcal{P})$ where $i(\mathcal{P})$ denotes the set of initial vertices of the paths in $\mathcal{P}$ and $t(\mathcal{P})$ is the set of terminal vertices of the paths in $\mathcal{P}$.

Proof. If D has $n = 1$ vertices then there is no path partition of D and the result holds vacuously, so we assume without loss of generality that D has $n \geq 2$ vertices. We give an induction proof and take as the induction hypothesis that the claim holds for all directed graphs on $n - 1$ vertices. By hypothesis, $t(\mathcal{P})$ is not a stable set, so there are vertices $y, z \in t(\mathcal{P})$ such that act (y, z) is an arc of D.

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Lemma 12.4 (continued 1)

Proof (continued). If vertex z is (trivial) path in $\mathcal{P}$, then we define $\mathcal{Q}$ to be the path partition by deleting this trivial path and extending the path in $\mathcal{P}$ that terminates at vertex y (such a path exists since $y \in t(\mathcal{P})$) by the arc (y, z) as shown in Figure 12.3 and the claim holds.

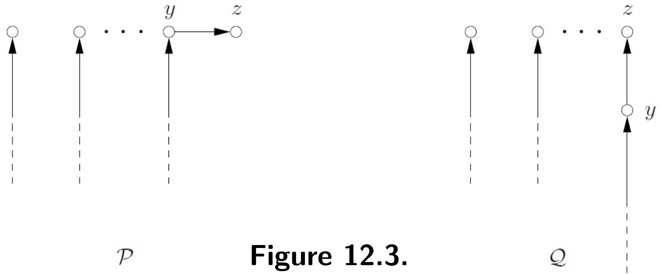

Figure 12.3.

So we may now assume without loss of generality that z is the terminal vertex of some nontrivial (directed) path P. Let x be the predecessor of z in path P.

Lemma 12.4 (continued 2)

Proof (continued). Define digraph $D' = D - z$, directed path $P' = P - z$, and path partition $\mathcal{P}'$ of D' as $\mathcal{P}' = (\mathcal{P} \setminus \{p\}) \cup P'$. That is, $\mathcal{P}'$ is the restriction of $\mathcal{P}$ to D' (see Figure 12.4).

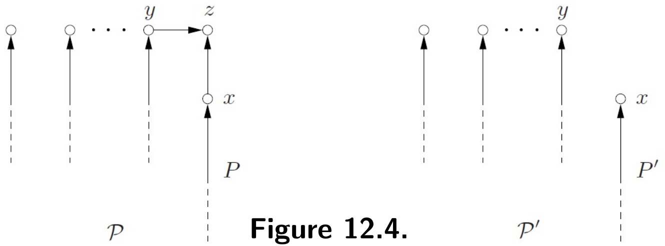

Figure 12.4.

There is no stable set in D' orthogonal to $\mathcal{P}'$, or else this stable set would be a stable set in D orthogonal to $\mathcal{P}$, contradicting the hypotheses. Notice that (see Figure 12.4 again) $t(\mathcal{P}') = t(\mathcal{P}(\setminus\{z\}) \cup \{x\}$ and $i(\mathcal{P}') - i(\mathcal{P})$.

Lemma 12.4 (continued 3)

Proof (continued). Now D' has $n-1$ vertices and so by the induction hypothesis there is a path partition $\mathcal{Q}'$ of D' such that $|\mathcal{Q}'| = |\mathcal{P}'| - 1$m $i(\mathcal{Q}') \subset i(\mathcal{P}')$, and $t(\mathcal{Q}') \subset t(\mathcal{P}\prime)$.

If $x \in t(\mathcal{Q}')$ then we define $\mathcal{Q}$ to be the path partition of D obtained from $\mathcal{Q}'$ by extending the path of $\mathcal{Q}'$ that terminates at x by the arc (x, z) (see Figure 12.5).

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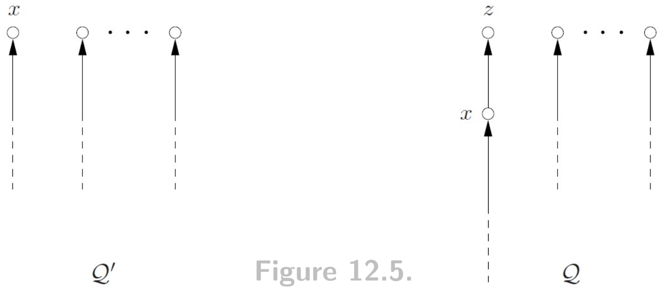

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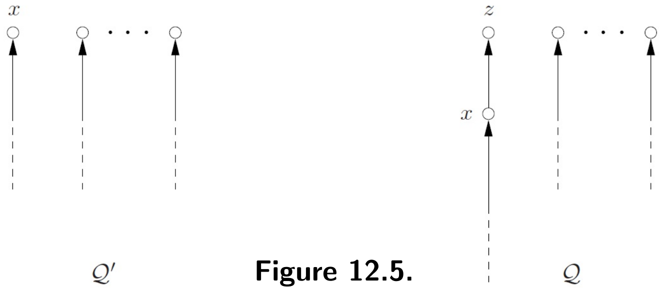

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Lemma 12.4 (continued 4)

Lemma 12.4. Let $\mathcal{P}$ be a path partition of a digraph D. Suppose that no stable set of D is orthogonal to $\mathcal{P}$. Then there is a path partition $\mathcal{Q}$ of D such that $|\mathcal{Q}| = |\mathcal{P}| - 1$, $i(\mathcal{Q}) \subset i(\mathcal{P})$, and $t(\mathcal{Q}) \subset t(\mathcal{P})$.

Proof (continued). Then $|\mathcal{Q}| = |\mathcal{P}| - 1$ (since $|\mathcal{P}'| = |\mathcal{P}|$ and $|\mathcal{Q}'| = |\mathcal{Q}|$, $i(\mathcal{Q} \subset i(\mathcal{P})$, and $t(\mathcal{Q}) \subset t(\mathcal{P})$ (in fact, the initial and terminal sets are equal; see Figures 12.4 and 12.5 again).

If $x \notin t(\mathcal{Q}')$, then $y \in t(\mathcal{Q}')$ (there is one less path in $\mathcal{Q}'$ than in $\mathcal{P}'$ and so there is one less terminal vertex in $t(\mathcal{Q}')$ than in $t(\mathcal{P}')$. so for $x \in t(\mathcal{P}')$ if $x \notin t(\mathcal{Q}')$ then all other elements of $t(\mathcal{P}') = (t(\mathcal{P}) \setminus \{z\}) \cup \{x\}$ must be in $t(\mathcal{Q}')$, including $y \in t(\mathcal{P})$). Define $\mathcal{Q}$ to be the path partition of D obtained from $\mathcal{Q}'$ by extending the path of $\mathcal{Q}'$ that terminates at y by the arc (y, z) of D. Again, $|\mathcal{Q}'| = |\mathcal{Q}|$, $i(\mathcal{Q}) \subset i(\mathcal{P})$, and $t(\mathcal{Q}) \subset t(\mathcal{P})$. So the induction step holds and by mathematical induction the result holds for all digraphs D on $n \geq 1$ vertices, as claimed. □

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Theorem 12.5. Dilworth's Theorem

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The minimum number of chains into which the elements of a partially ordered (finite) set P can be partitioned is equal to the maximum number of elements in an antichain of P.

Proof. Let the "poset" be $P = (X, \prec)$ where X is a finite set. Define digraph $D = D(P)$ with vertex set X and arcs (u, v) whenever $u \prec v$ in P. A chain $x_1 \prec x_2 \prec \cdots \prec x_n$ of P corresponds to a directed path from vertex x_1 to x_n in D. An antichain (of elements of X, no two of which are comparable) of P correspond to a stable set in D. No two elements of an antichain of P can belong to a common chain (because of transitivity of $\prec$), so the minimum number of chains in a chain partition is at least as large as the maximum number of elements in an antichain.

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Theorem 12.5. Dilworth's Theorem (continued)

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Proof (continued). Now the minimum number of paths in a path partition of D (which we have denoted as $\pi(D)$) is at least as large as the maximum number of elements in a stable set in D (which is the stability number of D, $\alpha(D)$); that is, $\pi \geq \alpha$. By the Gallai-Milgram Theorem $\pi \leq \alpha$. Therefore $\pi = \alpha$. That is, the minimum number of chains is a partitioning of P into chains is equal to the maximum size of an antichain of P, as claimed. □

Theorem 12.6. Richardson's Theorem

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Let D be a digraph which contains no directed odd cycle. Then S has a kernel.

Proof. If D has $n = 1$ vertex then the single vertex forms a kernel. If D has $n = 2$ vertices then it is bipartite and each individual vertex forms a kernel (unless D has no arcs, in which case the whole vertex set forms a kernel). We give an induction proof and take as the induction hypothesis that the claim holds for all directed graphs on less than n vertices.

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By Exercise 3.4.11, a strongly connected digraph which contains an odd cycle also contains a directed odd cycle. Since digraph D contains no directed off cycle, then it contains no odd cycle (more precisely, its underlying graph contains no off cycle). So by Theorem 4.7, D is bipartite. Each partite set of the bipartition is a kernel of D.

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Theorem 12.6. Richardson's Theorem (continued 1)

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Let D be a digraph which contains no directed odd cycle. Then S has a kernel.

Proof (continued). If D is not strongly connected, then let D_1 be a minimal strongly connected component of D (that is, one that dominates no other strongly connected component; by Exercise 3.4.6d(i), a minimal strongly connected component exists for every digraph D). Set $V_1 = V(D_1)$. Since D_1 is a proper subgraph of D (unless D is a complete digraph with $n > 2$ vertices, but then it contains a directed odd cycle in violation of the hypotheses). By the induction hypothesis, D_1 has a kernel S_1. Let V_2 be the set of vertices of D that dominate vertices of S_1, and define $D_2 = D - (V_1 \cup V_2)$. Again by induction, D_2 has a kernel S_2. Since S_1 is a kernel of D_1 then S_1 is a stable set in D_1; since S_2 is a kernel in D_2 then S_2 is a kernel of D_2. Since D_1 and D_2 share no vertices then $S = S_1 \cup S_2$ is a stable set in D.

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Theorem 12.6. Richardson's Theorem (continued 2)

Proof (continued). Since S_2 is a kernel in D_2, then each vertex in S_2 is dominated by some vertex in $V(D_2) - V(S_2) = V(D) - (V_1 \cup V_2) - V(S_2)$. By choice each vertex in S_1 by some vertex in V_2. So each vertex of $S_1 \cup S_2$ is dominated by some vertex in

$$\begin{aligned}(V(D) - (V_1 \cup V_2) - V(S_2)) \cup V_2 &= V(D) - V_1 - V(S_2) \\ &= V(D) - V(D_1) - V(S_2) \\ &\subset V(D) - V(S_1) - V(S_2) \\ &\quad \text{since } V(S_1) \subset V(D_1) \\ &= V(D) - V(S_1 \cup S_2)\end{aligned}$$

and so $S_1 \cup S_2$ is a kernel of D. Therefore the result holds for all V with n vertices. By mathematical induction, the result holds for all digraphs D, as claimed. □