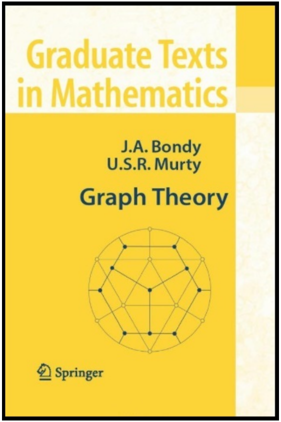
Graph Theory

Chapter 12. Stable Sets and Cliques

12.2. Turán's Theorem—Proofs of Theorems



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Let G be a simple graph which contains no K_k, where $k \geq 2$. Then $e(G) \leq e(T_{k-1,n})$, with equality if and only if $G \cong T_{k-1,n}$.

Proof. We give an induction proof on k. For $k = 2$, the hypothesis that G contains no K_k implies that G has no edges, so the inequality holds; the Turán graph $T_{1,n}$ is a "1-partite" with with n vertices and not edges, so that the equality holds. For the induction hypothesis, suppose the claim holds for all positive integers greater than or equal to 2 and less than k (so we are taking $k \geq 3$ now).

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Theorem 12.3. Turán's Theorem (continued)

Proof (continued). So by the induction hypothesis, $e(X) \le e(T_{k-2,\Delta})$ with equality if and only if $G[x] \cong T_{k-2,\Delta}$. Since $Y = V \setminus X$ then each edge of G incident with a vertex of Y belongs to either $E(X, Y)$ (when the edge is also incident to a vertex in X) or $E(Y)$ (when both ends of the edge are in $Y = V \setminus X$), then $e(X, Y) + e(Y) \le \Delta(n - \Delta)$ with equality if and only if Y is a stable set all members of which have degree Δ (by Exercise 12.2.A). So

$$e(G) = e(X) + e(X, Y) + e(Y) \le e(T_{k-2,\Delta}) + \Delta(n - \Delta),$$

and $e(G) \le e(H)$ where H is the graph obtained from a copy of $T_{k-2,\Delta}$ (on Δ vertices) by adding a stable set of $n - \Delta$ vertices and joining each vertex of this set to each vertex of $T_{k-2,\Delta}$. Observe that H is then a complete $(k-1)$-partite graph on $(n - \Delta) + \Delta = n$ vertices. By Exercise 1.1.11(a), $e(H) \le e(T_{k-1,n})$ with equality if and only if $H \cong T_{k-1,n}$. Therefore $e(G) \le e(H) \le e(T_{k-1,n})$ with equality if and only if $G \cong H \cong T_{k-1,n}$, as claimed. □

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Theorem 12.4

Theorem 12.4. Let S be a finite set of diameter one in the plane. Then the number of pairs of points of S whose distance is greater than $1/\sqrt{2}$ is at most $\lfloor n^2/3 \rfloor$, where $n = |S|$. Moreover, for each $n \geq 2$, there is a set of n points of diameter one in which exactly $\lfloor n^2/3 \rfloor$ pairs of points are at distance greater than $1/\sqrt{2}$.

Proof. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite set of points in the plane with diameter 1. Consider the graph G with vertex set S and edge set $\{x_i x_j \mid d(x_i, x_j) > 1/\sqrt{2}\}$, where $d(x_i, x_j)$ denotes the Euclidean distance between x_i and x_j as points in the Cartesian plane. We'll show that G cannot contain a copy of K_4. The convex hull determined by four points is either a line, a triangle, or a quadrilateral (see Figure 12.9).

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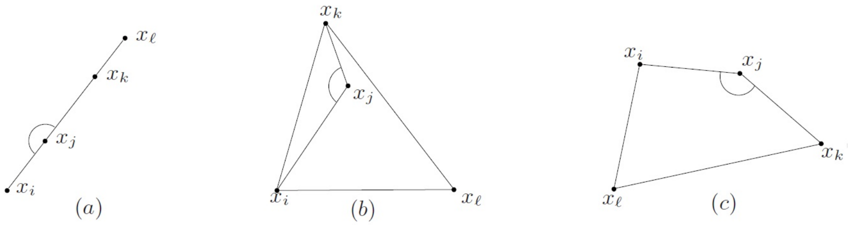
Proof (continued).



In each case, some three of the points, say $x_i x_j x_k$, form an angle $\widehat{x_i x_j x_k}$ of at least 90°. If $d(x_i, x_j) > 1/\sqrt{2}$ and $d(x_j, x_k) > 1/\sqrt{2}$, then by the Law of Cosines (since $90^\circ \le \widehat{x_i x_j x_k} \le 180^\circ$)

$$(d(x_i, x_k))^2 = (d(x_i, x_j))^2 + (d(x_j, x_k))^2 - 2d(x_i, x_j)d(x_j, x_k)\cos(\widehat{x_i x_j x_k})$$

$$\ge (d(x_i, x_j))^2 + (d(x_j, x_k))^2 > (1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1.$$

But this is a contradiction to the fact that the diameter of S is at most 1.

Theorem 12.4 (continued 2)

Proof (continued). So for any four points in S, at least two of the points cannot be adjacent in G (the points x_i and x_k, as labeled in Figure 12.9). Hence, G cannot contain a copy of K_2. By Turán's Theorem (Theorem 12.7) with $k = 4$, we have that $e(G) \leq e(T_{3,n})$. By Exercise 1.1.11, $e(T_{3,n}) = \lfloor n^2/3 \rfloor$ so that $e(G) \leq \lfloor n^2/3 \rfloor$, as claimed.

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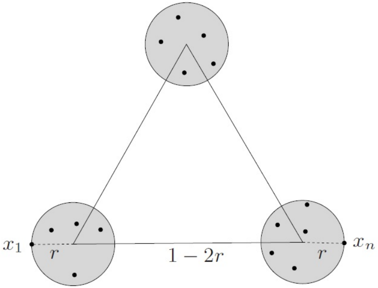
Next, we construct a set such that exactly $\lfloor n^2/3 \rfloor$ pairs of points are at a distance greater than $1/\sqrt{2}$ apart. Choose r such that $0 < r < (1 - 1/\sqrt{2})/4$. Consider three circles of radius r whose centers are $1 - 2r$ from one another. See Figure 12.10.



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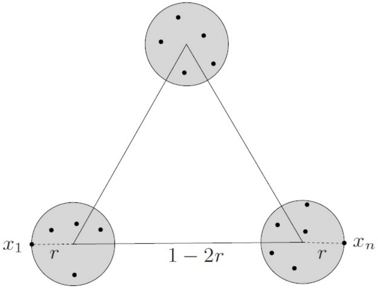

Figure 12.10

Theorem 12.4 (continued 3)

Theorem 12.4. Let S be a finite set of diameter one in the plane. Then the number of pairs of points of S whose distance is greater than $1/\sqrt{2}$ is at most $\lfloor n^2/3 \rfloor$, where $n = |S|$. Moreover, for each $n \geq 2$, there is a set of n points of diameter one in which exactly $\lfloor n^2/3 \rfloor$ pairs of points are at distance greater than $1/\sqrt{2}$.

Proof (continued). Set $p = \lfloor n/3 \rfloor$. Place points $x_1, x_2, \ldots, x_p$ in one circle, points $x_{p+1}, x_{p+2}, \ldots, x_{2p}$ in another, and points $x_{2p+1}, x_{2p+2}, \ldots, x_n$ in the third circle. So so in such a way that $d(x_1, x_n) = 1$ and $x_2, x_3, \ldots, x_{n-1}$ are in the interiors of their circles.Notice from the geometry that any two points are at most 1 unit apart (and the diameter of the set of points is 1). Two points in different circles are at a distance greater than $1 - 4r > 1 - (1 - 1/\sqrt{2}) = 1/\sqrt{2}$ apart (and only if the points are in different circles). There are $p^2 + 2p(n - 2p) = \lfloor n/3 \rfloor^2 + 2\lfloor n/3 \rfloor(\lceil n/3 \rceil) = \lfloor n^2/3 \rfloor$ pairs of points at least $1/\sqrt{2}$ apart, and hence $\lfloor n^2/3 \rfloor$ edges of G, as claimed. □