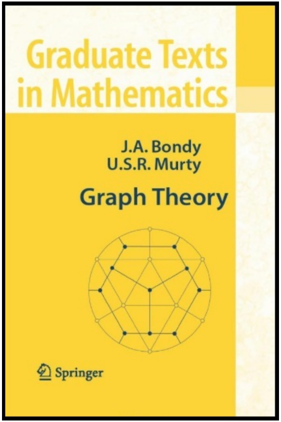
Graph Theory

Chapter 12. Stable Sets and Cliques

12.3. Ramsey's Theorem—Proofs of Theorems



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$$r(k,\ell) \leq r(k,\ell-1) + r(k-1,\ell).$$

Furthermore, if $r(k,\ell-1)$ and $f(k-1,\ell)$ are both even, strict inequality holds in the inequality.

Proof. Let G be a graph on $r(k,\ell-1) + r(k-1,\ell)$ vertices and let $v \in V$. We consider two cases:

1. Vertex v is nonadjacent to a set S of at least $r(k,\ell-1)$ vertices.
2. Vertex v is adjacent to a set T of at least $r(k-1,\ell)$ vertices.

Since G has $r(k,\ell-1) + r(k-1,\ell)$ vertices, then there are $r(k,\ell-1) + r(k-1,\ell) - 1$ vertices in G other than vertex v. So the number of vertices to which v is nonadjacent plus the number of vertices to which v is adjacent is equal to $r(k,\ell-1) + r(k-1,\ell) - 1$; hence, either Case 1 or Case 2 must hold.

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Theorem 12.9 (continued 1)

Proof (continued). In Case 1, the induced subgraph $G[S]$ of G contains either a clique of k vertices or a stable set of $\ell - 1$ vertices. Therefore, $G[S \cup \{v\}]$ contains either a clique of k vertices (since $G[S]$ does) or a stable set of ℓ vertices (the stable set of $\ell - 1$ vertices in $G[S]$ along with vertex v which is not adjacent to any vertices of S). Since $G[S \cup \{v\}]$ is a subgraph of G, then G also contains these sets. In Case 2, the induced subgraph $G[T]$ contains either a clique of $k - 1$ vertices or a stable set of ℓ vertices. Therefore, $G[T \cup \{v\}]$ contains either a clique of k vertices (the clique of $k - 1$ vertices in $G[T]$ along with vertex v which is adjacent to all vertices of T) or a stable set of ℓ vertices (since $G[T]$ does). Since $G[T \cup \{v\}]$ is a subgraph of G, then G also contains these sets. This proves the inequality.

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Theorem 12.9 (continued 2)

Proof (continued). Now suppose that $r(k, \ell - 1)$ and $r(k - 1, \ell)$ are both even, and let G' be a graph on $r(k, \ell - 1) + r(k - 1, \ell) - 1$ vertices. So G' has an odd number vertices, by Corollary 1.2 there is some vertex v' of G' of even degree. In particular, v' cannot be adjacent to precisely $r(k - 1, \ell) - 1$ vertices. So v' must be adjacent to at least vertices $r(k - 1, \ell)$ vertices (in which Case 2 above holds) or v' must be nonadjacent to at least $r(k, \ell - 1)$ vertices (in which Case 1 above holds). That is, in graph G' either Case 1 or Case 2 hold, and hence, as shown above, G' either contains a clique on k vertices or a stable set on ℓ vertices. So we have by the inequality established above (but with $r(k - 1, \ell)$ there replaced with $r(k - 1, \ell) - 1$ here) gives

$$r(k, \ell) \leq r(k, \ell - 1) + r(k - 1, \ell) - 1 < r(k, \ell - 1) + r(k - 1, \ell),$$

as claimed. □

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Theorem 12.10

Theorem 12.10. For all positive integers k and ℓ, $r(k,\ell) \leq \binom{k+\ell-2}{k-1}$.

Proof. We give an inductive proof on the sum $k+\ell$. If $k+\ell \leq 5$ then either k or ℓ must be less than 3. By Note 12.A, for $k+\ell \leq 5$ we have $r(1,\ell) = 1 \leq \binom{k+\ell-2}{k-1}$ since every combination is at least 1, $r(k,2) = k = \binom{k+\ell-2}{k-1} = \binom{k}{k-1} = k$, and similarly, by symmetry, $r(k,1)$ and $r(2,\ell)$ are also bounded as claimed. So we take $k+\ell \leq 5$ as the base case(s).

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Let m and n be positive integers and for the induction hypothesis suppose the theorem is valid for all integers k and ℓ such that $5 \leq k+\ell \leq m+n$.

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Proof (continued). Then

$$\begin{aligned}
r(m,n) &\leq r(m,n-1) + r(m-1,n) \text{ by Theorem 12.9} \\
&\leq \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2} \text{ by the induction hypothesis} \\
&= \frac{(m+n-3)!}{(n-2)!(m-1)!} + \frac{(m+n-3)!}{(n-1)!(m-2)!} \\
&= \frac{(m+n-3)!((n-1)+(m-1))}{(n-1)!(m-1)!} = \frac{(m+n-2)!}{(n-1)!(m-1)!} \\
&= \binom{m+n-2}{m-1}.
\end{aligned}$$

So the induction step is established. Therefore, by mathematical induction, the claim holds for all positive integers k and ℓ. □

Theorem 12.12

Theorem 12.12. For all positive integers k, $r(k,k) \geq 2^{k/2}$.

Proof. By Note 12.A, $r(1,1) = 1$ and $r(2,2) = 2$, so we just need to consider $k \geq 3$. Let $\mathcal{G}_n$ be the set of all simple graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{G}_n^k$ be the set of these labeled simple graphs which have a clique on k vertices. We have $\mathcal{G}_n| = 2^{\binom{n}{2}}$ (since for any of the $\binom{n}{2}$ pairs of vertices may or may not be joined by an edge; see Note 1.2.B).

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$$1 \cdot 2^{\binom{n-k}{2}} \cdot 2^{(n-k)k} = 2^{(n^2-n-k^2+k)/2} = 2^{\binom{n}{2}-\binom{k}{2}}$$

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Proof (continued). Because there are $\binom{n}{k}$ distinct k-element subsets of $\{v_1, v_2, \ldots, v_n\}$ we have $|\mathcal{G}_n^k| \leq \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}}$. We pick up an inequality here because there may be graphs in $\mathcal{G}_n^k$ which have more than one k-clique in which case they are counted once on the left-hand-side by the bound on the right-hand-side counts then more once. Therefore

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} \leq \frac{\binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}}}{2^{\binom{n}{2}}} = \binom{n}{k} 2^{-\binom{k}{2}}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} 2^{-\binom{k}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!}.$$

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Proof (continued). Suppose $n < 2^{k/2}$. Then, since $k \geq 3$,

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} < \frac{n^k 2^{-\binom{k}{2}}}{k!} < \frac{2^{k^2/2} 2^{-\binom{k}{2}}}{k!} = \frac{2^{k/2}}{k!} < \frac{1}{2}.$$

That is, if $n < 2^{k/2}$ then strictly fewer than half of the graphs in $\mathcal{G}_n$ contain a stable set of k vertices. By considering complements, we similarly have that strictly fewer than half of the graphs in $\mathcal{G}_n$ contain a stable set of k vertices. Therefore some graph in $\mathcal{G}_n$ contains neither a clique of k vertices nor a stable set of k vertices. That is, if $n < 2^{k/2}$ then there aren't necessarily enough vertices in a graph on n vertices to guarantee that the graph either contains a clique on k vertices or a stable set on k vertices. Hence, $n < r(k,k)$ and so we must have $r(k,k) \geq 2^{k/2}$ as claimed. □

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Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of the set of integers $\{1, 2, \ldots, r_n\}$ into n subsets. Then some A_i contains three integers x, y, and z satisfying the equation z.

Proof. Consider the complete graph whose vertex set is $\{1, 2, \ldots, r_n\}$. Color the edges of this graph with colors $1, 2, \ldots, n$ by the rule that the edge uv is assigned color i if $|u - v| \in A_i$. By the definition of this general Ramsey number $r_n = r(t_1, t_2, \ldots, t_n) = r(3, 3, \ldots, 3)$ we know that some A_j contains a K_3; that is, there are three vertices a, b, c such that edges ab, bc, and ac all have the same color j. Suppose, without loss of generality, that $a > b > c$. Let $x = a - b$, $y = b - c$, and $z = a - c$. Then, since ab, bc, ac are color j, then $x, y, z \in A_j$. Also, $x + y = (a - b) + (b - c) = a - c = z$, as claimed. □

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