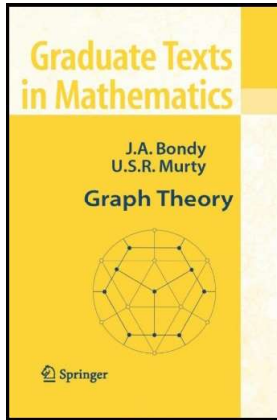


Graph Theory

Chapter 13. The Probabilistic Method

13.2. Expectation—Proofs of Theorems



Lemma 13.1

Lemma 13.1. THE CROSSING LEMMA.

Let G be a simple graph with $m \geq 4n$. Then $cr(G) \geq \frac{1}{64} \frac{m^3}{n^2}$.

Proof. Consider a planar embedding \tilde{G} of G with $cr(G)$ crossings. Let S be a random subset of V obtained by choosing each vertex of G independently with probability $p = 4n/m \leq 1$. Define the induced subgraph $H = G[S]$ and define the planar embedding $\tilde{H} = \tilde{G}[S]$ (a sub-embedding of \tilde{G}).

Define random variables X, Y, Z on $\Omega = \mathcal{P}(V)$ as follows: X is the number of vertices in S , Y is the number of edges in $H = G[S]$, and Z is the number of crossings of $\tilde{H} = \tilde{G}[S]$. By the “trivial lower bound” of Note 13.2.A, for any $S \in \Omega$ we have $Z(S) \geq Y(S) - 3X(S)$.

Lemma 13.1 (continued)

Lemma 13.1. THE CROSSING LEMMA.

Let G be a simple graph with $m \geq 4n$. Then $cr(G) \geq \frac{1}{64} \frac{m^3}{n^2}$.

Proof (continued). Now $E(X) = pn$, $E(Y) = p^2m$ (since the probability of a particular edge being chosen in the probability that both ends of the edge are chosen), and $E(Z) = p^4cr(G)$ (since each crossing in \tilde{G} is determined by two particular edges or their four particular vertices). Hence

$$p^4 cr(G) \geq p^2 m - 3pn$$

or

$$cr(G) \geq \frac{m}{p^2} - \frac{3n}{p^3} = \frac{pm - 3n}{p^3} = \frac{4n - 3n}{(4n/m)^3} = \frac{nm^3}{64n^3} = \frac{1}{64} \frac{m^3}{n^2},$$

as claimed. \square

Theorem 13.2

Theorem 13.2. Let P be a set of n points in the plane, and let ℓ be the number of lines in the plane passing through at least $k + 1$ of these points, where $1 \leq k \leq 2\sqrt{2n}$. Then $\ell < 32n^2/k^3$.

Proof. Form a graph G with vertex set P whose edges are the segments between consecutive points on the lines which pass through at least $k + 1$ points of set P . So each line through $k + 1$ points results in a path in G with k edges (and $k + 1$ vertices). Since there are ℓ such lines, then G has at least ℓk edges. Now the k edges of G which result from a particular line cannot cross each other and any two edges of G that cross must lie on two distinct lines in the plane and since two lines intersect in at most one point, we can only have at most one crossing associated with any pair of edges (that is, given any lines ℓ_1 and ℓ_2 in the plane, there can be at most one crossing of edges in G which are associated with these lines). So the number of crossings in G satisfies $cr(G) \leq \binom{\ell}{2}$.

Theorem 13.2 (continued)

Proof (continued). So either $kl < 4n$ (in which case The Crossing Lemma does not apply) or $\binom{\ell}{2} \geq \text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2} \geq \frac{1}{64} \frac{(kl)^3}{n^2}$ by The Crossing Lemma. If $kl < 4n$ then

$$\begin{aligned} \ell &< \frac{4n}{k} \leq \frac{4n}{k} \left(\frac{8n}{k^2} \right) \text{ since we hypothesize that} \\ 1 &\leq \frac{2\sqrt{2n}}{k} \text{ or } 1 \leq \frac{8n}{k^2} \\ &= 32n/k^3, \text{ as claimed.} \end{aligned}$$

If $\binom{\ell}{2} \geq \text{cr}(G) \geq \frac{1}{64} \frac{(kl)^3}{n^2}$ then we have

$$\begin{aligned} \frac{\ell^2}{2} > \frac{\ell^2 - \ell}{2} = \binom{\ell}{2} \geq \text{cr}(G) \geq \frac{1}{64} \frac{(kl)^3}{n^2} \text{ and so } \ell^2 > \frac{1}{32} \frac{(kl)^3}{n^2} \text{ or} \\ \ell &< \frac{32n^2}{k^3}, \text{ as claimed.} \quad \square \end{aligned}$$

Theorem 13.3

Theorem 13.3. Let P be a set of n points in the plane, and let k be the number of pairs of points of P at unit distance. Then $k < 5n^{4/3}$.

Proof. Draw a unit circle around each point of P . Let n_i be the number of these circles passing through exactly i points of P . Each circle passes through between 0 and $n-1$ points of P so the total number of circles, n , satisfies $n = \sum_{i=0}^{n-1} n_i$. For any pair of points p_1 and p_2 a distance 1 apart, there is a circle centered at p_1 passing through p_2 , and there is a circle centered at p_2 passing through p_1 . Now the number of points which lie on some circle, counting multiplicity, is $\sum_{i=0}^{n-1} in_i$, so the number k of pairs of points a distance 1 apart satisfies $k = \frac{1}{2} \sum_{i=0}^{n-1} in_i$.

Theorem 13.3 (continued 1)

Proof (continued). Now form a graph H with vertex set P whose edges are the arcs of the circles between consecutive points on the circles that pass through at least three points of P . Then

$$e(H) = \sum_{i=3}^{n-1} in_i = \sum_{i=0}^{n-1} in_i - (0n_0 + 1n_1 + 2n_2) = 2k - n_1 - 2n_2 \geq 2k - 2n.$$

Some pairs of vertices of H might be joined by two parallel edges. Delete from H one of each pair of parallel edges, so as to obtain a simple graph G with $e(G) \geq k - n$. Now any two circles in the plane can intersect in at most two points, and since we have n circles then the greatest possible number of intersections of circles is $2 \times \binom{n}{2} = n(n-1)$. Hence for graph G , $\text{cr}(G) \leq n(n-1)$. If $e(G) < 4n$ then $4n > e(G) \geq k - n$ and $k \leq 5n \leq 5n^{4/3}$, as claimed.

Theorem 13.3 (continued 2)

Theorem 13.3. Let P be a set of n points in the plane, and let k be the number of pairs of points of P at unit distance. Then $k < 5n^{4/3}$.

Proof (continued). If $e(G) \geq 4n$ then The Crossing Lemma applies so that

$$\text{cr}(G) \geq \frac{1}{64} \frac{(e(G))^3}{n^2} \geq \frac{(k-n)^3}{64n^2}.$$

Hence (for $n \geq 1$)

$$n^2 > n^2 - n = n(n-1) \geq \text{cr}(G) \geq \frac{(k-n)^3}{64n^2}$$

$$\text{or } n^{2/3} > \frac{k-n}{4n^{2/3}} \text{ or } 4n^{4/3} > k-n \text{ or}$$

$$k < 4n^{4/3} + n \leq 4n^{4/3} + n^{4/3} = 5n^{4/3}, \text{ as claimed.} \quad \square$$

Proposition 13.4

Proposition 13.4. MARKOV'S INEQUALITY

Let X be a nonnegative finite random variable on probability space (Ω, P) and $t > 0$. Then $P(X \geq t) \leq \frac{E(X)}{t}$.

Proof. We have by the definition of expectation that

$$\begin{aligned} E(X) &= \sum_{\omega \in \Omega} X(\omega)P(\omega) \geq \sum_{\omega \in \Omega, X(\omega) \geq t} X(\omega)P(\omega) \\ &\geq \sum_{\omega \in \Omega, X(\omega) \geq t} tP(\omega) = t \sum_{\omega \in \Omega, X(\omega) \geq t} P(\omega) = tP(X \geq t), \end{aligned}$$

as claimed. \square

Corollary 13.5

Corollary 13.5. Let X_n be a nonnegative integer-valued random variable in a probability space (Ω_n, P_n) where $n \in \mathbb{N}$. If $E(X_n) \rightarrow 0$ as $n \rightarrow \infty$, then $P(X_n = 0) \rightarrow 1$ as $n \rightarrow \infty$.

Proof. Let $X = X_n$ and $t = 1$ in Markov's Inequality (Proposition 13.4). Then $P(X_n \geq 1) \leq E(X_n)$ for all $n \in \mathbb{N}$. Since we hypothesize $E(X_n) \rightarrow 0$ as $n \rightarrow \infty$, then it follows (by the Sandwich Theorem, say) that $P(X_n \geq 1) \rightarrow 0$ as $n \rightarrow \infty$. Since $P(X_n = 0) = 1 - P(X_n \geq 1)$ then $P(X_n = 0) \rightarrow 1$ as $n \rightarrow \infty$, as claimed. \square

Theorem 13.6

Theorem 13.6. A random graph in $\mathcal{G}_{n,p}$ almost surely has stability number at most $\lceil 2p^{-1} \log n \rceil$.

Proof. Let $G \in \mathcal{G}_{n,p}$ and let S be a given set of $k+1$ vertices of G where $k \in \mathbb{N}$. Then S is a stable set of G if none of the $\binom{k+1}{2}$ edges joining two vertices of set S are present in G . This happens with probability $(1-p)^{\binom{k+1}{2}}$.

Let A_S denote the event that S is a stable set of G , and let X_S denote the indicator random variable for this event. By equation (13.5) we have

$$E(X_S) = P(X_S = 1) = P(A_S) = (1-p)^{\binom{k+1}{2}}.$$

Let X_n be the number of stable sets of cardinality $k+1$ in $G \in \mathcal{G}_{n,p}$. Since X_S is the indicator random variable then $X_n = \sum_{S \subseteq V, |S|=k+1} X_S$.

Theorem 13.6 (continued 1)

Proof (continued). By linearity of expectation,

$$\begin{aligned} E(X_n) &= E\left(\sum_{S \subseteq V, |S|=k+1} X_S\right) = \sum_{S \subseteq V, |S|=k+1} E(X_S) \\ &= \sum_{S \subseteq V, |S|=k+1} (1-p)^{\binom{k+1}{2}} = \binom{n}{k+1} (1-p)^{\binom{k+1}{2}}. \end{aligned}$$

In Exercise 13.2.1 it is to be shown that

$$\binom{n}{k+1} \leq \frac{n^{k+1}}{(k+1)!} \text{ and } 1-p \leq e^{-p}.$$

So

$$\begin{aligned} E(X) &= \binom{n}{k+1} (1-p)^{\binom{k+1}{2}} \leq \frac{n^{k+1}}{(k+1)!} (e^{-p})^{\binom{k+1}{2}} \\ &= \frac{n^{k+1} e^{-p(k+1)k/2}}{(k+1)!} = \frac{(ne^{-pk/2})^{k+1}}{(k+1)!}. \quad (13.7) \end{aligned}$$

Theorem 13.6 (continued 2)

Proof (continued). Suppose $k = \lceil 2p^{-1} \log n \rceil$. Then $k \geq 2p^{-1} \log n$ and so $e^k \geq e^{2p^{-1} \log n} = n^{2/p}$ or $1 \geq n^{2/p} e^{-k}$ or $1^{p/2} \geq (n^{2/p} e^{-k})^{p/2}$ or $1 \geq n e^{-pk/2}$. So by equation (13.7), $E(X_n) \leq \frac{1}{(k+1)!}$. As $n \rightarrow \infty$, $k \geq \frac{2}{p} \log n \rightarrow \infty$ and hence $E(X_n) \rightarrow 0$ (this conclusion also holds for any value of k larger than $\lceil 2p^{-1} \log n \rceil$). Now X_n is integer valued and nonnegative, so the hypotheses of Corollary 13.5 are satisfied and hence $P(X_n = 0) \rightarrow 1$ as $n \rightarrow \infty$. Since X_n is the number of stable sets of cardinality $k+1 = \lceil 2p^{-1} \log n \rceil + 1$ in G , we have shown that, almost surely, there are 0 such stable sets. So, equivalently, almost surely the maximum size stable set (that is, the stability number) is of cardinality at most $k = \lceil 2p^{-1} \log n \rceil$, as claimed. \square