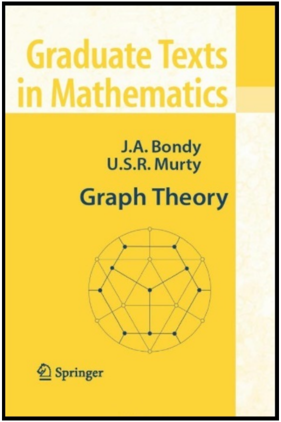
Graph Theory

Chapter 13. The Probabilistic Method

13.2. Expectation—Proofs of Theorems



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Lemma 13.1

Lemma 13.1. THE CROSSING LEMMA.
Let G be a simple graph with $m \geq 4n$. Then $\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}$.

Proof. Consider a planar embedding $\tilde{G}$ of G with $\mathrm{cr}(G)$ crossings. Let S be a random subset of V obtained by choosing each vertex of G independently with probability $p = 4n/m \leq 1$. Define the induced subgraph $H = G[S]$ and define the planar embedding $\tilde{H} = \tilde{G}[S]$ (a sub-embedding of $\tilde{G}$).

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Define random variables X, Y, Z on $\Omega = \mathcal{P}(V)$ as follows: X is the number of vertices in S, Y is the number of edges in $H = G[S]$, and Z is the number of crossings of $\tilde{H} = \tilde{G}[S]$. By the "trivial lower bound" of Note 13.2.A, for any $S \in \Omega$ we have $Z(S) \geq Y(S) - 3X(S)$.

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Lemma 13.1 (continued)

Lemma 13.1. The Crossing Lemma.

Let G be a simple graph with $m \geq 4n$. Then $\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}$.

Proof (continued). Now $E(X) = pn$, $E(Y) = p^2m$ (since the probability of a particular edge being chosen in the probability that both ends of the edge are chosen), and $E(Z) = p^4\mathrm{cr}(G)$ (since each crossing in $\tilde{G}$ is determined by two particular edges or their four particular vertices). Hence

$$p^4\mathrm{cr}(G) \geq p^2m - 3pn$$

or

$$\mathrm{cr}(G) \geq \frac{m}{p^2} - \frac{3n}{p^3} = \frac{pm - 3n}{p^3} = \frac{4n - 3n}{(4n/m)^3} = \frac{nm^3}{64n^3} = \frac{1}{64}\frac{m^3}{n^2},$$

as claimed. □

Theorem 13.2

Theorem 13.2. Let P be a set of n points in the plane, and let ℓ be the number of lines in the plane passing through at least $k+1$ of these points, where $1 \leq k \leq 2\sqrt{2n}$. Then $\ell < 32n^2/k^3$.

Proof. Form a graph G with vertex set P whose edges are the segments between consecutive points on the lines which pass through at least $k+1$ points of set P. So each line though $k+1$ points results in a path in G with k edges (and $k+1$ vertices). Since there are ℓ such lines, then G has at least ℓk edges.

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Theorem 13.2 (continued)

Proof (continued). So either $k\ell < 4n$ (in which case The Crossing Lemma does not apply) or $\binom{\ell}{2} \geq \text{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2} \geq \frac{1}{64}\frac{(k\ell)^3}{n^2}$ by The Crossing Lemma. If $k\ell < 4n$ then

$$\begin{aligned} \ell &< \frac{4n}{k} \leq \frac{4n}{k}\left(\frac{8n}{k^2}\right) \text{ since we hypothesize that} \\ &\quad 1 \leq \frac{2\sqrt{2n}}{k} \text{ or } 1 \leq \frac{8n}{k^2} \\ &= 32n/k^3, \text{ as claimed.} \end{aligned}$$

If $\binom{\ell}{2} \geq \text{cr}(G) \geq \frac{1}{64}\frac{(k\ell)^3}{n^2}$ then we have $\frac{\ell^2}{2} > \frac{\ell^2 - \ell}{2} = \binom{\ell}{2} \geq \text{cr}(G) \geq \frac{1}{64}\frac{(k\ell)^3}{n^2}$ and so $\ell^2 > \frac{1}{32}\frac{(k\ell)^3}{n^2}$ or $\ell < \frac{32n^2}{k^3}$, as claimed. □

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Theorem 13.3

Theorem 13.3. Let P be a set of n points in the plane, and let k be the number of pairs of points of P at unit distance. Then $k < 5n^{4/3}$.

Proof. Draw a unit circle around each point of P. Let n_i be the number of these circles passing through exactly i points of P. Each circle passes through between 0 and $n-1$ points of P so the total number of circles, n, satisfies $n = \sum_{i=0}^{n-1} n_i$.

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Theorem 13.3 (continued 1)

Proof (continued). Now form a graph H with vertex set P whose edges are the arcs of the circles between consecutive points on the circles that pass through at least three points of P. Then

$$e(H) = \sum_{i=3}^{n-1} in_i = \sum_{i=0}^{n-1} in_i - (0n_0 + 1n_1 + 2n_2) = 2k - n_1 - 2n_2 \geq 2k - 2n.$$

Some pairs of vertices of H might be joined by two parallel edges. Delete from H one of each pair of parallel edges, so as to obtain a simple graph G with $e(G) \geq k - n$. Now any two circles in the plane can intersect in at most two points, and since we have n circles then the greatest possible number of intersections of circles is $2 \times \binom{n}{2} = n(n-1)$. Hence for graph G, $\mathrm{cr}(G) \leq n(n-1)$. If $e(G) < 4n$ then $4n > e(G) \geq k - n$ and $k \leq 5n \leq 5n^{4/3}$, as claimed.

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Theorem 13.3 (continued 2)

Theorem 13.3. Let P be a set of n points in the plane, and let k be the number of pairs of points of P at unit distance. Then $k < 5n^{4/3}$.

Proof (continued). If $e(G) \geq 4n$ then The Crossing Lemma applies so that

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{(e(G))^3}{n^2} \geq \frac{(k-n)^3}{64n^2}.$$

Hence (for $n \geq 1$)

$$n^2 > n^2 - n = n(n-1) \geq \mathrm{cr}(G) \geq \frac{(k-n)^3}{64n^2}$$

or $n^{2/3} > \dfrac{k-n}{4n^{2/3}}$ or $4n^{4/3} > k - n$ or
$k < 4n^{4/3} + n \leq 4n^{4/3} + n^{4/3} = 5n^{4/3}$, as claimed. □

Proposition 13.4

Proposition 13.4. MARKOV'S INEQUALITY
Let X be a nonnegative finite random variable on probability space (Ω, P) and $t > 0$. Then $P(X \geq t) \leq \dfrac{E(X)}{t}$.

Proof. We have by the definition of expectation that

$$E(X) = \sum_{\omega \in \Omega} X(\omega)P(\omega) \geq \sum_{\omega \in \Omega, X(\omega) \geq t} X(\omega)P(\omega)$$

$$\geq \sum_{\omega \in \Omega, X(\omega) \geq t} tP(\omega) = t \sum_{\omega \in \Omega, X(\omega) \geq t} P(\omega) = tP(X \geq t),$$

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Corollary 13.5

Corollary 13.5. Let X_n be a nonnegative integer-valued random variable in a probability space (Ω_n, P_n) where $n \in \mathbb{N}$. If $E(X_n) \to 0$ as $n \to \infty$, then $P(X_n = 0) \to 1$ as $n \to \infty$.

Proof. Let $X = X_n$ and $t = 1$ in Markov's Inequality (Proposition 13.4). Then $P(X_n \geq 1) \leq E(X_n)$ for all $n \in \mathbb{N}$. Since we hypothesize $E(X_n) \to 0$ as $n \to \infty$, then it follows (by the Sandwich Theorem, say) that $P(X_n \geq 1) \to 0$ as $n \to \infty$. Since $P(X_n = 0) = 1 - P(X_n \geq 1)$ then $P(X_n = 0) \to 1$ as $n \to \infty$, as claimed. □

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Theorem 13.6

Theorem 13.6. A random graph in $\mathcal{G}_{n,p}$ almost surely has stability number at most $\lceil 2p^{-1} \log n \rceil$.

Proof. Let $G \in \mathcal{G}_{n,p}$ and let S be a given set of $k+1$ vertices of G where $k \in \mathbb{N}$. Then S is a stable set of G if none of the $\binom{k+1}{2}$ edges joining two vertices of set S are present in G. This happens with probability $(1-p)^{\binom{k+1}{2}}$.

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Let A_s denote the event that S is a stable set of G, and let X_S denote the indicator random variable for this event. By equation (13.5) we have

$$E(X_S) = P(X_S = 1) = P(A_S) = (1-p)^{\binom{k+1}{2}}.$$

Let X_n be the number of stable sets of cardinality $k+1$ in $G \in \mathcal{G}_{n,p}$. Since X_S is the indicator random variable then $X_n = \sum_{S \subseteq V, |S| = k+1} X_S$.

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Theorem 13.6 (continued 1)

Proof (continued). By linearity of expectation,

$$E(X_n) = E\left(\sum_{S \subseteq V, |S|=k+1} X_S\right) = \sum_{S \subseteq V, |S|=k+1} E(X_S)$$

$$= \sum_{S \subseteq V, |S|=k+1} (1-p)^{\binom{k+1}{2}} = \binom{n}{k+1}(1-p)^{\binom{k+1}{2}}.$$

In Exercise 13.2.1 it is to be shown that

$$\binom{n}{k+1} \leq \frac{n^{k+1}}{(k+1)!} \text{ and } 1-p \leq e^{-p}.$$

So

$$E(X) = \binom{n}{k+1}(1-p)^{\binom{k+1}{2}} \leq \frac{n^{k+1}}{(k+1)!}(e^{-p})^{\binom{k+1}{2}}$$

$$= \frac{n^{k+1}e^{-p(k+1)k/2}}{(k+1)!} = \frac{(ne^{-pk/2})^{k+1}}{(k+1)!}. \qquad (13.7)$$

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Theorem 13.6 (continued 2)

Proof (continued). Suppose $k = \lceil 2p^{-1} \log n \rceil$. Then $k \geq 2p^{-1} \log n$ and so $e^k \geq e^{2p^{-1} \log n} = n^{2/p}$ or $1 \geq n^{2/p} e^{-k}$ or $1^{p/2} \geq \left(n^{2/p} e^{-k}\right)^{p/2}$ or $1 \geq ne^{-pk/2}$. So by equation (13.7), $E(X_n) \leq \dfrac{1}{(k+1)!}$. As $n \to \infty$, $k \geq \dfrac{2}{p} \log n \to \infty$ and hence $E(X_n) \to 0$ (this conclusion also holds for any value of k larger than $\lceil 2p^{-1} \log n \rceil$). Now X_n is integer valued and nonnegative, so the hypotheses of Corollary 13.5 are satisfied and hence $P(X_n = 0) \to 1$ as $n \to \infty$. Since X_n is the number of stable sets of cardinality $k + 1 = \lceil 2p^{-1} \log n \rceil + 1$ in G, we have shown that, almost surely, there are 0 such stable sets. So, equivalently, almost surely the maximum size stable set (that is, the stability number) is of cardinality at most $k = \lceil 2p^{-1} \log n \rceil$, as claimed. □

Theorem 13.6 (continued 2)

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