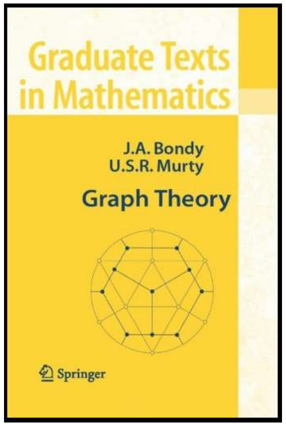
Graph Theory

Chapter 13. The Probabilistic Method

13.3. Variance—Proofs of Theorems



Theorem 13.7

Theorem 13.7. Chebyshev's Inequality.
Let X be a random variable on a finite probability space and let $t > 0$. Then

$$P(|X - E(X)| \geq t) \leq \frac{V(X)}{t^2}.$$

Proof. We have

$$\begin{aligned} P(|X - E(X)| \geq t) &= P((X - E(X))^2 \geq t^2) \\ &\leq \frac{E((X - E(X))^2)}{t^2} \text{ by Markov's Inequality} \\ &\quad \text{(Proposition 13.4)} \\ &= \frac{V(X)}{t^2}, \end{aligned}$$

as claimed. □

Corollary 13.8

Corollary 13.8. Let X_n be a random variable in a finite probability space (Ω_n, P_n) where $n \geq 1$. If $E(X_n) \neq 0$ and $V(X_n) \ll E^2(X_n)$, then $P(X_n = 0) \to 0$ as $n \to \infty$.

Proof. With $X = X_n$ and $t = |E(X_n)|$, Chebyshev's Inequality implies

$$P(|X_n - E(X_n)| \geq |E(X_n)|) \leq \frac{V(X_n)}{E^2(X_n)}. \qquad (*)$$

Now when $X_n = 0$ we have $|X_n - E(X_n)| = |0 - E(X_n)| = |E(X_n))|$, and "$X_n = 0$" is included in values of X_n such that $|X_n - E(X_n)| \geq |E(X_n)|$. Hence, $P(X_n = 0) \leq P(|X_n - E(X_n)| \geq |E(X_n)|)$ and so, by $(*)$, $P(X_n = 0) \leq V(X)/E^2(X_n)$. The hypothesis $V(X_n) \ll E^2(X_n)$ means that $V(X_n)/E^2(X_n) \to 0$ as $n \to \infty$. Therefore (by the Sandwich Theorem, say) $P(X_n = 0) \to 0$ as $n \to \infty$, as claimed. □

Theorem 13.9

Theorem 13.9. Let $G \in \mathcal{G}_{n,1/2}$. For $0 \leq k \leq n$, set $f(k) = \binom{n}{k} 2^{-\binom{k}{2}}$ and let k^* be the least value of k for which $f(k)$ is less than one. Then almost surely the stability number of G, $\alpha(G)$, takes one of the three values $k^* - 2$, $k^* - 1$, or k^*.

Proof. Let $G \in \mathcal{G}_{n,1/2}$ and let $X \subset V$. Define X_S as the indicator random variable for the event A_S that S is a stable set in G. Set $X = \sum_{S \subseteq V, |S| = k} X_S$ (so X is the number of stable sets of cardinality k in G). As shown in the proof of Theorem 13.6, $E(X) = \binom{n}{k} 2^{-\binom{k}{2}}$ (replace the $k+1$ in the proof of Theorem 13.6 with k here and take p of Theorem 13.6 as $1/2$ here), so we have $E(X) = f(k) \neq 0$ and, as is to be shown in Exercise 13.2.11(b), almost surely $\alpha(G) \leq k^*$. So if we show also that almost surely $\alpha(G) \geq k^* - 2$, the result will follow.

Theorem 13.9 (continued 1)

Proof (continued). We set $k = k^* - 2$ and show $V(X) \ll E^2(X)$. We can then apply Corollary 13.8 to conclude that $P(X = 0) \to 0$ as $n \to \infty$. That is, G almost surely has a stable set of size $k = k^* - 2$ and so almost surely $\alpha(G) \geq k^* - 2$. Now we establish that fact that $V(X) \ll E^2(X)$.

By Exercise 13.2.11(b) we have for $k = k^* - 2$ that

$$k < 2\log_2 n \text{ and } f(k) \geq n/4. \qquad (13.9)$$

It is to be shown in Exercise 13.3.1 that

$$V(X) \leq E(X) + \sum_{S \neq T} C(X_S, C_T). \qquad (*)$$

Let S and T be two sets of k vertices. If $|S \cap T| \in \{0, 1\}$ then $C(X_S, X_T) = 0$ since no edge of G has both ends in $S \cap T$ and the events of S and T being stable sets are independent so that $E(X_S X_T) = E(X_S)E(X_T)$.

Theorem 13.9 (continued 2)

Proof (continued). If $|S \cap T| = i$ where $2 \leq i \leq k-1$ then, with $\overline{A}$ denoting the complement of event A,

$$C(X_S, X_T) = E(X_S X_T) - E(X_S)E(X_T) \leq E(X_S X_T)$$

$$= 0P(\overline{A}_S \cap \overline{A}_T) + 0P(A_S \cap \overline{A}_T) + 0P(\overline{A}_S \cap A_T) + 1P(A_S \cap A_T) = P(A_S \cap A_T). \ (**)$$

Now in event A_S, S is a stable set if G contains none of the possible $\binom{k}{2}$ edges with both ends in S, and since $p = 1/2$ then $P(A_S) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$. Similarly, $P(A_T) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$. Now both A_S and A_T are stable sets if G contains none of the possible edges with either (1) both ends in S, or (2) both ends in T. When $|S \cap T| = i$, this involves a total of $2\binom{k}{2} - \binom{i}{2}$ edges so that

$$P(A_S \cap A_T) = \left(\frac{1}{2}\right)^{2\binom{k}{2} - \binom{i}{2}} = 2^{\binom{i}{2} - 2\binom{k}{2}}.$$

Theorem 13.9 (continued 3)

Proof (continued). We now count the number of choices of S and T. First, there are $\binom{n}{k}$ choices for S, then $\binom{k}{i}$ choices for $S \cap T$, and finally $\binom{n-k}{k-i}$ choices for the vertices in $T \setminus S$. So there are $\binom{n}{k}\binom{k}{i}\binom{n-k}{k-i}$ choices for S and T. Then

$$\begin{aligned} V(X) &\leq E(X) + \sum_{S \neq T} C(X_S, X_T) \text{ by } (*) \\ &\leq E(X) + \sum_{S \neq T} P(A_S \cap A_T) \text{ by } (**) \\ &= E(X) + \sum_{i=2}^{k-1} \binom{n}{k}\binom{k}{i}\binom{n-k}{k-i} 2^{\binom{i}{2} - 2\binom{k}{2}}. \end{aligned}$$

Now $E(X) = \binom{n}{k} 2^{-\binom{k}{2}} \ll E^2(X) = \binom{n}{k}^2 2^{-2\binom{k}{2}}$ since

$$\frac{E(X)}{E^2(X)} = \frac{1}{\binom{n}{k} 2^{-\binom{k}{2}}} = \frac{2^{\binom{k}{2}}}{\binom{n}{k}} \to 0 \text{ as } n \to \infty.$$

Theorem 13.9 (continued 4)

Proof (continued). So it remains to show that

$$\sum_{i=2}^{k-1} \binom{n}{k}\binom{k}{i}\binom{n-k}{k-i} 2^{\binom{i}{2} - 2\binom{k}{2}} \ll E^2(X) = \binom{n}{k}^2 2^{-2\binom{k}{2}}$$

or equivalently that

$$\binom{n}{k}^{-1} \sum_{i=2}^{k-1} g(i) \to 0 \text{ as } n \to \infty \qquad (13.10)$$

where $g(i) = \binom{k}{i}\binom{n-k}{k-i} 2^{\binom{i}{2}}$. We have

$$g(2) = \binom{k}{2}\binom{n-k}{k-2} 2 = k(k-1)\binom{n-k}{k-2} < k^2 \binom{n}{k-2}.$$

Theorem 13.9 (continued 5)

Proof (continued). For $2 \le i \le k-2$,

$$\frac{g(i+1)}{g(i)} = \frac{\binom{k}{i+1}\binom{n-k}{k-i-1}2^{\binom{i+1}{2}}}{\binom{k}{i}\binom{n-k}{k-i}2^{\binom{i}{2}}} = \frac{\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k)!}{(k-i-1)!(n-2k+i+1)!}}{\frac{k!}{i!(k-i)!}\frac{(n-k)!}{(k-i)!(n-2k+i)!}}\frac{2^{(i+1)i/2}}{2^{i(i-1)/2}}$$

$$= \frac{(k-i)}{(i+1)}\frac{(k-i)}{(n-2k+i+1)}2^i < \frac{k^2 2^i}{i(n-2k)} \text{ since } i+1>i \text{ and } n-2k+i+1>n-2k.$$

Set $t = \lfloor c \log_2 n \rfloor$ where $0 < c < 1$. Observe that $f(x) = 2^x/x$ is an increasing function for $x > 1/\ln 2 \approx 1.44$.

Theorem 13.9 (continued 6)

Proof (continued). Then for $2 \le i \le t-1$ we have

$$\frac{g(i+1)}{g(i)} < \left(\frac{2^i}{i}\right)\left(\frac{k^2}{n-2k}\right) \text{ as just shown}$$

$$\le \left(\frac{2^t}{t}\right)\left(\frac{k^2}{n-2k}\right) \text{ since } 2^x/x \text{ is increasing and } i \le t-1 < t$$

$$< \left(\frac{n^c}{c\log_2 n}\right)\left(\frac{(2\log_2 n)^2}{n-4\log_2 n}\right) \text{ since } 2^x/x \text{ is increasing and } t = \lfloor c\log_2 n\rfloor \le c\log_2 n, \text{ and } k < 2\log_2 n \text{ by (13.9)}$$

$$= \frac{4n^c\log_2 n}{c(n-4\log_2 n)}.$$

Now $\frac{4n^c\log_2 n}{c(n-4\log_2 n)} \to 0$ so for n sufficiently large $\frac{4n^c\log_2 n}{c(n-4\log_2 n)} \le 1$, and hence $\frac{g(i+1)}{g(i)} \le 1$. That is, for $2 \le i \le t-1$ and n sufficiently large, $g(i+1) \le g(i)$. Therefore $g(t) \le g(t-1) \le \cdots \le g(2)$.

Theorem 13.9 (continued 7)

Proof (continued). So

$$\binom{n}{k}^{-1}\sum_{i=2}^{t} g(i) \le \binom{n}{k}^{-1} tg(2) < \binom{n}{k}^{-1} tk^2\binom{n}{k-2}$$

$$= \frac{k!(n-k)!}{n!}tk^2\frac{n!}{(k-2)!(n-k+2)!} = \frac{tk^2k(k-1)}{(n-k+2)(n-k+1)}$$

$$= \frac{tk^4-tk^3}{n^2-n(k-1)-n(k-2)+(k-2)(k-1)}$$

$$= \frac{tk^4-tk^3}{n^2-n(2k-3)+(k-2)(k-1)} \to 0 \text{ as } n \to \infty. \qquad (13.11)$$

In equation (13.10) we consider $\sum_{t=2}^{k-1} g(i)$, so we now need to address the values of i of $t+1, t+2, \ldots, k-1$.

Theorem 13.9 (continued 8)

Proof (continued). ...we now need to address the values of i of $t+1, t+2, \ldots, k-1$. We have

$$\sum_{i=t+1}^{k-1} g(i) = \sum_{i=t+1}^{k-1}\binom{k}{i}\binom{n-k}{k-i}2^{\binom{i}{2}}$$

$$= 2^{\binom{k}{2}}\sum_{i=t+1}^{k-1}\binom{k}{k-i}\binom{n-k}{k-i}2^{\binom{i}{2}-\binom{k}{2}} \text{ since } \binom{k}{i} = \binom{k}{k-i}$$

$$= 2^{\binom{k}{2}}\sum_{i=t+1}^{k-1}\binom{k}{k-i}\binom{n-k}{k-i}2^{-(k-i)(k+i-1)/2} \text{ since}$$

$$\binom{i}{2}-\binom{k}{2} = \frac{i(i-1)}{2}-\frac{k(k-1)}{2} = \frac{i^2-i-k^2+k}{2}$$

$$= \frac{-k(k-i)-k(i-1)+i(i-1)}{2}\ldots$$

Theorem 13.9 (continued 9)

Proof (continued). ...

$$\sum_{i=t+1}^{k-1} g(i) = 2^{\binom{k}{2}} \sum_{i=t+1}^{k-1} \binom{k}{k-i}\binom{n-k}{k-i} 2^{-(k-i)(k+i-1)/2} \text{ since}$$

$$\binom{i}{2} - \binom{k}{2} = \cdots = \frac{-k(k-i)-(k-i)(i-1)}{2} = \frac{-(k-1)(k+i-1)}{2}$$

$$= 2^{\binom{k}{2}} \sum_{j=1}^{k-t-1} \binom{k}{j}\binom{n-k}{j} 2^{-j(2k-j-1)/2} \text{ by letting } j = i-t$$

and when i ranges from $t+1$ to $k-1$ then j ranges from 1 to $k-t-1$, respectively, and $k-i$ ranges from $k-t-1$ to 1, respectively; also replacing j with $k-i$ in $2k-j-1$ gives $2k-(k-i)-1 = k+i-1$, as given

Theorem 13.9 (continued 10)

Proof (continued). ...

$$\sum_{i=t+1}^{k-1} g(i) = 2^{\binom{k}{2}} \sum_{j=1}^{k-t-1} \binom{k}{j}\binom{n-k}{j} 2^{-j(2k-j-1)/2}$$

$$= 2^{\binom{k}{2}} \sum_{j=1}^{k-t-1} \frac{k!}{j!(k-j)!}\frac{(n-k)!}{j!(n-k-j)!}\left(2^{-(2k-j-1)/2}\right)^j$$

$$< 2^{\binom{k}{2}} \sum_{j=1}^{k-t-1} \frac{k!}{(k-j)!}\frac{(n-k)!}{(n-k-j)!}\left(2^{-(k+t)/2}\right)^j \text{ since for}$$

$1 \le j \le k-t-1$ we have $2k-j-1 \ge 2k-(k-t-1)-1 = k+t$

$$< 2^{\binom{k}{2}} \sum_{j=1}^{k-t-1} \left(k(n-k)2^{-(k+t)/2}\right)^j. \qquad (\dagger\dagger)$$

Theorem 13.9 (continued 11)

Proof (continued). To bound this last term, we use the fact that $k^* + \log_2 k^* - 1 \ge 2\log_2 n$, as is to be shown in Exercise 13.2.11(a). This implies $2^{k^*+\log_2 k^*-1} \ge 2^{2\log_2 n}$ or $2^{k^*}k^*2^{-1} \ge n^2$ or $2^{k^*/2}\sqrt{k^*}1/\sqrt{2} \ge n$ or $2^{-k^*/2} \le \sqrt{k^*/2}n^{-1}$ or $2^{-(k+2)/2} \le \sqrt{(k+2)/2}n^{-1}$ (since $k = k^*-2$) or $2^{-k/2} \le 2\sqrt{(k+2)/2}n^{-1} = \sqrt{2k+4}n^{-1}$. So for n sufficiently large

$$k(n-k)2^{-(k+t)/2} \le k(n-k)\sqrt{2k+4}n^{-1}2^{-t/2}$$

$$< k(n-k)\sqrt{2k+4}n^{-1}\sqrt{2}n^{-c/2}$$

since $t = \lfloor c\log_2 n \rfloor > (c\log_2 n) - 1$ and so $2^t > 2^{(c\log_2 n)-1} = n^c/2$ or $2^{-t/2} < \sqrt{2}n^{-c/2}$

$$= k\sqrt{4k+8}\left(1-\frac{k}{n}\right)n^{-c/2} \le 1$$

for n sufficiently large (since this quantity goes to 0 as $n \to \infty$). (This differs slightly from the book's computations...the book may have a small error in it here.)

Theorem 13.9 (continued 12)

Proof (continued). So from $(\dagger\dagger)$, for n sufficiently large,

$$\sum_{i=t+1}^{k-1} g(i) < 2^{\binom{k}{2}} \sum_{j=1}^{k-t-1}\left(k(n-k)2^{-(k+t)/2}\right)^j \le 2^{\binom{k}{2}} \sum_{j=1}^{k-t-1} 1 = 2^{\binom{k}{2}}(k-t-1).$$

From equation (13.9) we have $f(k) = \binom{n}{k}2^{-\binom{k}{n}} \ge n/4$, so we now have

$$\binom{n}{k}^{-1} \sum_{i=t+1}^{k-1} g(i) < \binom{n}{k}^{-1} 2^{\binom{k}{2}}(k-t-1)$$

$$= \frac{k-t-1}{f(k)} \le \frac{k-t-1}{n/4} = \frac{4(k-t-1)}{n}$$

and as $n \to \infty$ we see that

$$\binom{n}{k}^{-1} \sum_{i=t+1}^{k-1} g(i) \to 0 \text{ as } n \to \infty. \qquad (13.12)$$

Theorem 13.9 (continued 13)

Theorem 13.9. Let $G \in \mathcal{G}_{n,1/2}$. For $0 \leq k \leq n$, set $f(k) = \binom{n}{k} 2^{-\binom{k}{2}}$ and let k^* be the least value of k for which $f(k)$ is less than one. Then almost surely the stability number of G, $\alpha(G)$, takes one of the three values $k^* - 2$, $k^* - 1$, or k^*.

Proof (continued). Combining (13.11) and (13.12) we have (13.10):

$$\binom{n}{k}^{-1} \sum_{i=2}^{k-1} g(i) \to 0 \text{ as } n \to \infty.$$

As described above, this is sufficient to establish the claim. □