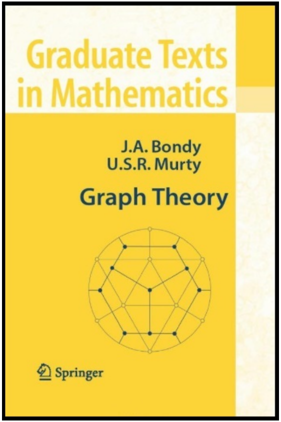
Graph Theory

Chapter 13. The Probabilistic Method

13.4. Evolution of Random Graphs—Proofs of Theorems



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Proof. Let $g \in \mathcal{G}_{n,p}$. For each k-subset $S \subseteq V$, let A_S be the event that the induced subgraph $G[S]$ contains a copy of F, and let X_S be the indicator random variable of A_S. Set random variable $X = \sum_{S \subseteq V, |S|=k} X_S$, so that X is the number of k-subsets which span copies of F. Notice that X is then no greater than the total number of copies of F in G.

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We first bound the expectation of X. Consider a k-subset $S \subseteq V$. If $G[S]$ contains a copy of F, there is a bijection $f : V(F) \to S$ (since $|V(F)| = |S| = k$) such that $f(u)f(v)$ is an edge of $G[S]$ whenever uv is an edge of F (but not necessarily conversely). So for a given bijection $f : V(F) \to S$, the probability that all these l edges $f(u)f(v)$ are present in $G[S]$ is p^l.

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Theorem 13.11 (continued 1)

Proof (continued). Thus $E(X_S) = p(A_S) \geq p^l$ (greater than or equal to because there is more then on bijection). Because there are $k!$ bijections $f : V(F) \to S$, ther are $k!$ possible copies of F in $G[S]$. So $E(X_S) = P(A_S) \leq k!p^l$ (less than or equal to because copies of F in $G[S]$ may have edges in common and so are not independent). We have

$$\begin{aligned} \frac{n^k p^l}{k^k} &\leq \binom{n}{k} p^l \text{ by Exercise 13.2.1(a)} \\ &\leq \sum_{S \subseteq V, |S|=k} E(X_S) \text{ since } E(X_S) \geq p^l \text{ and there} \\ &\quad \text{are } \binom{n}{k} \text{ } k\text{-subsets of } V \\ &= E\left(\sum_{S \subseteq V, |S|=k} X_S\right) \text{ by the linearity of expectation} \ldots \end{aligned}$$

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Theorem 13.11 (continued 2)

Proof (continued).

$$\begin{aligned}\frac{n^k p^l}{k^k} &\leq E(X) \leq \binom{n}{K} k!p^l \text{ since } E(X_S) \leq k!p^l \text{ and there} \\ &\quad \text{are } \binom{n}{k} \text{ } k\text{-subsets of } V \\ &\leq n^k p^l \text{ by Exercise 13.2.1(a).} \qquad (13.13)\end{aligned}$$

So if $p \ll n^{-k/l}$ (that is, if $p/n^{-kl} = pn^{k/l} \to 0$ as $n \to \infty$, and hence $p^l n^k \to 0$ as $n \to \infty$) then $E(X) \leq n^k p^l \to 0$ as $n \to \infty$. By Markov's Inequality (Proposition 13.4), for any $t > 0$ we ave $P(X \geq t) \leq E(X)/t$ and so $P(X \geq t) \to 0$ as $n \to \infty$. Hence $P(X = 0) \to 1$ as $n \to \infty$ and G almost surely contains no copy of F.

We now bound the variance of X. By Exercise 13.3.1

$$V(X) \leq E(X) + \sum_{S \neq T} C(X_S, X_T). \qquad (13.8)$$

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Theorem 13.11 (continued 3)

Proof (continued). As in the proof of Theorem 3.9, the value of the covariance $C(X_S, X_T)$ depends only on $|S \cap T|$. If $|S \cap T| \in \{0, 1\}$ then $C(X_S, X_T) = 0$ since no edge can be shared by $G[S]$ and $G[T]$ and the events A_S and A_T are independent. If $|S \cap T| = i$, where $2 \leq i \leq k - 1$, then each copy F_S of F in $G[S]$ meets each copy F_T of F in $G[T]$ in i vertices. Because F is balanced, the average degrees of a subgraph of F does not exceed $2e(F)/v(F) = 2l/k$. Now $F_S \cap F_T$ is a subgraph of F on i vertices, so the sum of the degrees of these vertices is at most $2il/k$ and hence $F_S \cap F_T$ consists of at most il/k edges. Hence the graph $F_S \cup F_T$ then has at least $2l - il/k$ edges. So the probability that both F_S and F_T are present in G is $p^{2l-il/k}$.

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$$\begin{aligned} C(X_S, X_T) &= E(X_S X_T) - E(X_S)E(X_T) \leq E(X_S X_T) \\ &= P(A_S \cap A_T) \text{ since } X_S \text{ and } X_T \text{ are indicator random} \\ &\quad \text{variables; see } (**) \text{ in the proof of Theorem 13.9} \ldots \end{aligned}$$

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Theorem 13.11 (continued 4)

Proof (continued). ...

$$C(X_S, X_T) = P(A_S \cap A_T) \leq (k!)^2 2^{2l - il/k}.$$

We now count the number of choices of S and T where $|S \cap T| = i$. First, there are $\binom{n}{k}$ choices for S, then $\binom{k}{i}$ choices for $S \cap T$, and finally $\binom{n-k}{k-i}$ choices for the vertices of $T \setminus S$. So there are $\binom{n}{k}\binom{k}{i}\binom{n-k}{k-i}$ choices for S and T (as in the proof of Theorem 13.9). But we are only interested in the size of $S \cap T$, not the precise elements of $S \cap T$. So there are $\binom{n}{k}\binom{n-k}{k-i}$ pairs (S, T) of k-subsets with $|S \cap T| = i$.

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$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1} \cdots \frac{n-k+1}{1} \leq n^k$$

and, similarly,
$\binom{n-k}{k-i} = \frac{(n-k)!}{(k-i)!(n-2k-i)!} = \frac{(n-k)}{(k-i)}\frac{(n-k-1)}{(k-i-1)} \cdots \frac{(n-2k-i+1)}{1} \leq n^{k-i}$ then. . .

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Theorem 13.11 (continued 5)

Proof (continued).

$$\begin{aligned}
\sum_{S \neq T} C(X_S, X_T) &\leq \sum_{S \neq T} E(X_S X_T) = \sum_{S \neq T} P(A_S \cap A_T) \\
&\leq \sum_{i=2}^{k-1} \binom{n}{k} \binom{n-k}{k-i} (k!)^2 p^{2i - il/k} \\
&\leq \sum_{i=2}^{k-1} n^k n^{k-i} (k!)^2 p^{2l - il/k} \text{ since } \binom{n}{k} \leq n^k \\
&\quad \text{and } \binom{n-k}{k-i} \leq n^{k-i} \\
&= (k!)^2 \sum_{i=2}^{k-1} n^{2k} n^{-i} p^{2l} p^{-il/k} \\
&= (k!)^2 \sum_{i=1}^{k-1} (n^k p^l)^2 (np^{l/k})^{-i}. \qquad (13.14)
\end{aligned}$$

Theorem 13.11 (continued 6)

Proof (continued). If $p \gg n^{-k/l}$ (that is, if $p/n^{-k/l} = pn^{k/l} \to \infty$ as $n \to \infty$, and hence $np^{l/k} \to \infty$ as $n \to \infty$) then $(np^{l/k})^{-1} \to 0$ for $i \geq 1$. By equation (13.13), $\frac{n^k p^l}{k^k} = \left(\frac{np^{l/k}}{k}\right)^k \leq E(X)$ and since $np^{l/k} \to \infty$ as $n \to \infty$ then $\left(\frac{np^{l/k}}{k}\right)^k \to \infty$ as $n \to \infty$ so that $E(X) \to \infty$ as $n \to \infty$. Therefore $E(X) \ll E^2(X)$ since $\frac{E(X)}{E^2(X)} = \frac{1}{E(X)} \to 0$ as $n \to \infty$. We now have

$$\begin{aligned} V(X) &\leq E(X) + \sum_{S \neq T} C(X_S, X_T) \text{ by Exercise 13.3.1 (equation (13.8))} \\ &\leq E(X) + (k!)^2 \sum_{i=2}^{k-1} (n^k p^l)^2 (np^{l/k})^{-1} \text{ by equation (13.14)} \end{aligned}$$

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Proof (continued). If $p \gg n^{-k/l}$ (that is, if $p/n^{-k/l} = pn^{k/l} \to \infty$ as $n \to \infty$, and hence $np^{l/k} \to \infty$ as $n \to \infty$) then $(np^{l/k})^{-1} \to 0$ for $i \geq 1$. By equation (13.13), $\dfrac{n^k p^l}{k^k} = \left(\dfrac{np^{l/k}}{k}\right)^k \leq E(X)$ and since $np^{l/k} \to \infty$ as $n \to \infty$ then $\left(\dfrac{np^{l/k}}{k}\right)^k \to \infty$ as $n \to \infty$ so that $E(X) \to \infty$ as $n \to \infty$. Therefore $E(X) \ll E^2(X)$ since $\dfrac{E(X)}{E^2(X)} = \dfrac{1}{E(X)} \to 0$ as $n \to \infty$. We now have

$$\begin{aligned} V(X) &\leq E(X) + \sum_{S \neq T} C(X_S, X_T) \text{ by Exercise 13.3.1 (equation (13.8))} \\ &\leq E(X) + (k!)^2 \sum_{i=2}^{k-1} (n^k p^l)^2 (np^{l/k})^{-1} \text{ by equation (13.14)} \end{aligned}$$

Theorem 13.11 (continued 7)

Proof (continued).

$$\begin{aligned} V(X) &\le E(X) + (k!)^2 \sum_{i=2}^{k-1} (n^k p^l)^2 (np^{l/k})^{-1} \\ &\le E(X) + (k!)^2 \sum_{i=2}^{k-1} (np^{l/k})^{-1} k^{2k} E^2(X) \text{ since } \frac{n^k p^l}{k^k} \le E(X) \\ &\quad \text{and hence } (nkp^l)^2 \le k^{2k} E^2(X) \\ &= E(X) + E^2(X)(k!)^2 k^{2k} \sum_{i=2}^{k-1} (np^{l/k})^{-1}. \end{aligned}$$

Now

$$\frac{E^2(X)(k!)^2 k^{2k} \sum_{i=2}^{k-1} (np^{l/k})^{-i}}{E^2(X)} = (k!)^2 k^{2k} \sum_{i=2}^{k-1} (np^{l/k})^{-1} \longrightarrow 0$$

as $n \longrightarrow \infty$ so that...

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Proof (continued).

$$\begin{aligned} V(X) &\leq E(X) + (k!)^2 \sum_{i=2}^{k-1} (n^k p^l)^2 (np^{l/k})^{-1} \\ &\leq E(X) + (k!)^2 \sum_{i=2}^{k-1} (np^{l/k})^{-1} k^{2k} E^2(X) \text{ since } \frac{n^k p^l}{k^k} \leq E(X) \\ &\qquad \text{and hence } (nkp^l)^2 \leq k^{2k} E^2(X) \\ &= E(X) + E^2(X)(k!)^2 k^{2k} \sum_{i=2}^{k-1} (np^{l/k})^{-1}. \end{aligned}$$

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as $n \longrightarrow \infty$ so that...

Theorem 13.11 (continued 8)

Theorem 13.11. Let F be a nonempty balanced graph with k vertices and l edges. Then $n^{-k/l}$ is a threshold function for the property of containing F as a subgraph.

Proof (continued). ...so that

$$V(X) \leq E(X) + \sum_{S \neq T} C(X_S, X_T) \ll E^2(X).$$

So the hypotheses of Corollary 13.8 are satisfied and hence $P(X = 0) \to 0$ as $n \to \infty$. That is, graph G almost surely contains a copy of F. Therefore $n^{-k/l}$ is a threshold function for the property of containing F as a subgraph, as claimed. □