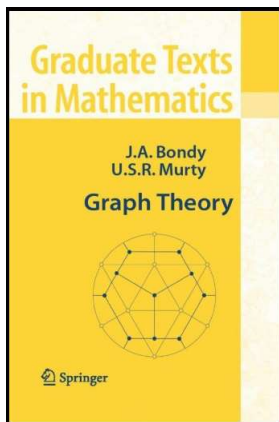


Graph Theory

Chapter 13. The Probabilistic Method 13.5. The Local Lemma—Proofs of Theorems



Theorem 13.12

Theorem 13.12. THE LOCAL LEMMA.

Let A_i , where $i \in N$, be events in a finite probability space (Ω, P) and let $N_i \subseteq N$ where $i \in N$. Suppose that, for all $i \in N$,

- (i) A_i is independent of the set of events $\{A_j \mid j \in N_i\}$,
- (ii) for each $i \in N$, there is a constant p_i where $0 < p_i < 1$, and for each $i \in N$ we have $P(A_i) = p_i \prod_{j \in N_i} (1 - p_j)$.

Set $B_i = \bar{A}_i$ where $i \in N$. Then, for any two disjoint subsets $R, S \subseteq N$,

$$P(B_R \cap B_S) \geq P(B_R) \prod_{i \in S} (1 - p_i). \quad (13.15)$$

In particular, when $R = \emptyset$ and $S = N$,

$$P\left(\bigcap_{i \in N} \bar{A}_i\right) \geq \prod_{i \in N} (1 - p_i) > 0. \quad (13.16)$$

Theorem 13.12 (continued 1)

Proof. If $S = \emptyset$ then $B_S = \bigcap_{i \in S} B_i = \bigcap_{i \in S} \bar{A}_i = \Omega$ (we could take this as the intersection of no sets) and $\prod_{i \in S} (1 - p_i) = 1$ (similarly, this could be taken as the definition of a product of no numbers), so

$$P(B_R \cap B_S) = P(B_R \cap \Omega) = P(B_R) = P(B_R)(1) \geq P(B_R) \prod_{i \in S} (1 - p_i),$$

and equation (13.15) holds when $S = \emptyset$.

If $|S| = 1$ and $S = \{i\}$, then $B_S = B_i$ and $\prod_{j \in S} (1 - p_j) = 1 - p_i$. Define $R_1 = R \setminus N_i$ and $S_1 = R \cap N_i$ (so that $R = \bar{R}_1 \cup S_1$). Then

$$\begin{aligned} P(A_i \cap B_R) &\leq P(A_i \cap B_{R_1}) \text{ since } R_1 \subseteq R \text{ and so} \\ &\quad B_R = \bigcap_{i \in R} \bar{A}_i \subseteq \bigcap_{i \in R_1} \bar{A}_i = B_{R_1} \text{ and } A_i \cap B_R \subseteq A_i \cap B_{R_1} \\ &= P(A_i)P(B_{R_1}) \text{ since } A_i \text{ is independent of} \\ &\quad \{A_j \mid j \notin N_i\} \supseteq R_1 \text{ by hypothesis (i).} \quad (*) \end{aligned}$$

Theorem 13.12 (continued 2)

Proof (continued). Since $S_1 \subseteq N_i \subseteq N$, then by hypothesis (ii)

$$P(A_i) \leq p_i \prod_{j \in N_i} (1 - p_j) \leq p_i \prod_{j \in S_1} (1 - p_j) \quad (**)$$

for some p_i with $0 < p_i < 1$ and some p_j with $0 < p_j < 1$. In Exercise 13.5.A(i) it is to be shown by induction on $|S_1|$ that

$$P(B_{R_1}) \prod_{j \in S_1} (1 - p_j) \leq P(B_{R_1} \cap B_{S_1}). \quad (\dagger)$$

Therefore

$$\begin{aligned} P(A_i \cap B_R) &\leq P(A_i)P(B_{R_1}) \text{ by } (*) \\ &\leq P(B_{R_1})p_i \prod_{j \in S_1} (1 - p_j) \text{ by } (**) \\ &\leq p_i P(B_{R_1} \cap B_{S_1}) \text{ by } (\dagger), \dots \quad (\dagger\dagger) \end{aligned}$$

Theorem 13.12 (continued 3)

Proof (continued). and so

$$\begin{aligned}
 P(B_R \cap B_S) &= P(B_R \cap B_i) \text{ since } S = \{i\} \\
 &= P(B_R) - P(B_R \cap A_i) \text{ since} \\
 &\quad B_R = (B_R \cap A_i) \cup (B_R \cap \bar{A}_i) = (B_R \cap A_i) \cup B_R \cap B_i \\
 &\geq P(B_R) - p_i P(B_R) \text{ by } (\dagger\dagger) \\
 &= P(B_R)(1 - p_i) = P(B_R) \prod_{i \in S} (1 - p_i),
 \end{aligned}$$

and equation (13.15) holds when $|S| = 1$.

If $|S| \geq 2$, then let R_1 and S_1 be nonempty disjoint sets which partition S so that $S = R_1 \cup S_1$. Then

$$\begin{aligned}
 P(B_R \cap B_S) &= P(B_R \cap B_{R_1 \cup S_1}) \\
 &= P(B_R \cap B_{R_1} \cap B_{S_1}) \text{ by definition of } B \text{'s as intersections} \\
 &= P(B_{R \cup R_1} \cap B_{S_1}) \text{ similarly.}
 \end{aligned}$$

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Theorem 13.12 (continued 4)

Proof (continued). In Exercise 13.5.A(ii) it is to be shown by induction on $|S_1|$ that

$$\begin{aligned}
 P(B_{R \cup R_1} \cap B_{S_1}) &\geq P(B_{R \cup R_1}) \prod_{i \in S_1} (1 - p_i) \\
 &= P(B_R \cap B_{R_1}) \prod_{i \in S_1} (1 - p_i). \quad (\dagger)
 \end{aligned}$$

In Exercise 13.5.A(iii) it is to be shown by induction on $|R \cup R_1|$ that

$$P(B_R \cap B_{R_1}) \geq P(B_R) \prod_{i \in R_1} (1 - p_i). \quad (\dagger\dagger)$$

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Theorem 13.12 (continued 5)

Proof (continued).

Therefore

$$\begin{aligned}
 P(B_R \cap B_S) &= P(B_{R \cup R_1} \cap B_{S_1}) \\
 &\geq P(B_R \cap B_{R_1}) \prod_{i \in S_1} (1 - p_i) \text{ by } (\dagger) \\
 &\geq P(B_R) \prod_{i \in R_1} (1 - p_i) \prod_{i \in S_1} (1 - p_i) \text{ by } (\dagger\dagger) \\
 &= P(B_R) \prod_{i \in S} (1 - p_i) \text{ since } S = R_1 \cup S_1,
 \end{aligned}$$

so equation (13.15) holds when $|S| \geq 2$ and hence holds for all S and R subsets of N , as claimed. \square

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Theorem 13.14

Theorem 13.14. THE LOCAL LEMMA—SYMMETRIC VERSION. let A_i , where $i \in N$, be events in a finite probability space (Ω, P) having a dependency graph with maximum degree d . Suppose $P(A_i) < 1/(e(d+1))$ for all $i \in N$ (where “ e ” here is the base of the natural log function). Then $P(\bigcap_{i \in N} \bar{A}_i) > 0$.

Proof. Set $p_1 = 1/(d+1) = p$ for $i \in N$ (this value of p maximizes the function $f(p) = p(1-p)^d$ for $p \in (0, 1)$ and will give us a “uniform bound” on $p(A_i)$ in hypothesis (ii) of The Local Lemma). Now the sets N_i are defined from the dependency graph (N_i includes all neighbors of vertex i in the dependency graph, so we have event A_i is independent of the events $\{A_j \mid j \notin N_i\}$, as required by hypothesis (i) of The Local Lemma).

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Theorem 13.14 (continued)

Proof (continued). Now

$$\begin{aligned}
 P(A_i) &\leq \frac{1}{e(d+1)} \text{ by hypothesis} \\
 &\leq \left(\frac{d}{d+1}\right)^d \left(\frac{1}{d+1}\right) \text{ since } 1 + \frac{1}{d} \leq e^{1/d} \text{ by Exercise 13.2.1(b)} \\
 &\quad \text{with } x = 1/d, \text{ or } (1 + 1/d)^d \leq e \text{ or } \left(\frac{d+1}{d}\right)^d \leq e \\
 &\quad \text{or } \left(\frac{d}{d+1}\right)^d \geq \frac{1}{e} \\
 &= p \prod_{j \in N_i} (1-p)^d \text{ since } p_j = p = \frac{1}{d+1} \text{ for all } j \in N_i.
 \end{aligned}$$

So hypothesis (ii) of the Local Lemma holds. Hence, by the Local Lemma (Theorem 13.12, the “in particular” part), $P(\bigcap_{i \in N} \bar{A}_i) > 0$, as claimed. \square

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Theorem 13.15

Theorem 13.15. Let $H = (V, \mathcal{F})$ be a hypergraph in which each edge has at least k elements and meets at most d other edges. If $e(d+1) \leq 2^{k-1}$ (again, “ e ” here is the base of the natural log function), then H is 2-colourable.

Proof. Consider a random 2-colouring of V , where each vertex receives one of two colours with probability $1/2$. For each edge F of H , denote by A_F the event that F is monochromatic. Then events A_F and A_G are independent unless edges F and G share vertices. Since an edge of H meets at most d other edges, then the dependence graph for the events $\{A_F \mid F \in \mathcal{F}\}$ has maximum degree d . Then $P(A_F) \leq 2 \cdot 1/2^k = 2^{1-k}$ (less than or equal since F has at least k elements, and times 2 because there are 2 colours). So $P(A_F) \leq 2^{1-k} \leq 1/(e(d+1))$ and the hypotheses of the symmetric version of The Local Lemma are satisfied. Hence, $P(\bigcap_{F \in \mathcal{F}} \bar{A}_F) > 0$ and, by Note 13.5.A, hence H is 2-colourable. \square

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Corollary 13.16

Corollary 13.16. Let $H = (V, \mathcal{F})$ be a k -uniform k -regular hypergraph, where $k \geq 9$. Then H is 2-colourable.

Proof. Since H is k -uniform, then each edge contains exactly k elements (i.e., vertices), and since H is k -regular then each vertex of H lies on k edges. So for given edge F of H , F contains k vertices and each lies on $k-1$ edges (along with edge F), so that edge F meets at most $d = k(k-1)$ other edges. If $k \geq 9$ then $e(d+1) = e(k(k-1)+1) \leq 2^{k-1}$ since (1) for $k=9$ we have $e(k(k-1)+1) = 73e \approx 198.4$, $2^{k-1} = 256$, and (2) $f(x) = 2^{x-1} - e(x^2 - x + 1)$ has derivative $f'(x) = (\ln 2)2^{x-1} - e(2x-1) > 0$ for $x \geq 9$, so that f is increasing for $x \geq 9$ and hence $f(x) \geq f(9) > 0$ for all $x \geq 9$. So the hypotheses of Theorem 13.15 are satisfied and hence H is 2-colourable. \square

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Theorem 13.17

Theorem 13.17. Let D be a strict (i.e., “simple”) k -diregular digraph where $k \geq 8$. Then D contains a directed even cycle.

Proof. Consider a random 2-colouring of V , where each vertex receives one of the two colours with probability $1/2$. For each vertex v of D , denote by A_v the event that $c(u) = c(v)$ for all $u \in N^+(v)$ (that is, A_v denotes the event that all outneighbors of v are the same colour as v). So A_v is independent of all A_u such that $(\{u\} \cup N^+(u)) \cap N^+(v) = \emptyset$ (that is, the outneighbors of v do not include any outneighbors of u nor u itself). Then A_v is dependent on some A_u when one (or more) of the k outneighbors of v is one of the k outneighbors of u or u itself. Since D is k -diregular, then each of the k -outneighbors of v can be an outneighbor of $k-1$ other vertices (along with being an outneighbor of v itself) and is a vertex associated with some A_u , so that each outneighbor of v is associated with up to k other events dependent on A_v . So in the dependency graph of the event $\{A_v \mid v \in V\}$, each vertex is of degree at most $d = k^2$.

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Theorem 13.17 (continued 1)

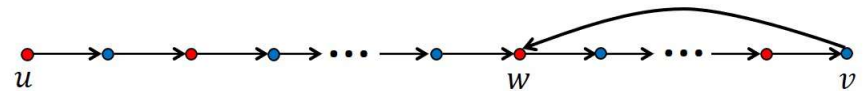
Theorem 13.17. Let D be a strict (i.e., “simple”) k -diregular digraph where $k \geq 8$. Then D contains a directed even cycle.

Proof (continued). Next, notice that \bar{A}_v is the event that v has an outneighbor of a different colour. Since v has k outneighbors then $P(A_v) = 1/2^k$. With $d = k^2$ we have $P(A_i) = \frac{1}{2^k} \leq \frac{1}{e(d+1)} = \frac{1}{e(k^2+1)}$ for $k \geq 8$, since (1) for $k = 8$ we have $\frac{1}{2^8} = \frac{1}{256} \approx 0.0039$, $\frac{1}{e(k^2+1)} = \frac{1}{65e} \approx 0.0057$, and (2) $f(x) = \frac{1}{e(x^2+1)} - 2^{-x}$ has derivative $f'(x) = \frac{-2x}{e(x^2+1)^2} + (\ln 2)2^{-x} > 0$ for $k \geq 8$ so that f is increasing for $x \geq 8$ and hence $f(x) \geq f(8) > 0$ for all $x \geq 8$. So the hypotheses of Theorem 3.14 are satisfied and hence $P(\cap_{v \in V} \bar{A}_v) > 0$. That is, there is a 2-colouring of V such that every $v \in V$ has an outneighbor of the opposite colour.

Theorem 13.17 (continued 2)

Theorem 13.17. Let D be a strict (i.e., “simple”) k -diregular digraph where $k \geq 8$. Then D contains a directed even cycle.

Proof (continued). With respect to this colouring, let uPv be a maximal (length) properly 2-coloured directed path in D and let w be an outneighbor of v of the opposite colour of v . Since uPv is maximal, then w must be some vertex of uPv (or else $uPvw$ would be a longer properly 2-coloured path in D). So take the segment of P from w to v (denoted wPv) and then add the arc from v to w to produce a cycle in D . This cycle is properly coloured and so is an even length cycle, as claimed. \square



Lemma 13.18

Lemma 13.18. Let $G = (V, E)$ be a simple graph and let $\{V_1, V_2, \dots, V_k\}$ be a partition of V into k sets, each of cardinality at least $2e\Delta$ (again, “e” here is the base of the natural log function). Then there is a stable set S in G such that $|S \cap V_i| = 1$ for $1 \leq i \leq k$.

Proof. By deleting vertices from G if necessary, we may assume that $|V_i| = t = \lceil 2e\Delta \rceil$ for $1 \leq i \leq k$ (we’ll show the existence of a stable set S under these conditions, then the deleted vertices and relevant edges can be added back in to G and this won’t have an effect on the stability of set S nor on the intersection property of set S). We select one vertex v_i at random from each set V_i for $1 \leq i \leq k$, and set $S = \{v_1, v_2, \dots, v_n\}$.

For an edge f of G , let A_f denote the event that both ends of f belong to S . Since $|V_i| = t$ for each i then $P(A_f) = 1/t^2$ for all $f \in E$. In Exercise 13.5.A is to be shown that A_f is dependent only on those events A_g such that an end of g lies in the same set V_i as an end of f .

Lemma 13.18 (continued 1)

Lemma 13.18. Let $G = (V, E)$ be a simple graph and let $\{V_1, V_2, \dots, V_k\}$ be a partition of V into k sets, each of cardinality at least $2e\Delta$ (again, “e” here is the base of the natural log function). Then there is a stable set S in G such that $|S \cap V_i| = 1$ for $1 \leq i \leq k$.

Proof (continued). For a given A_f where f has its ends in V_i and V_j (where possibly $i = j$), there are at most $t\Delta$ edges with an end in V_i and at most $t\Delta$ edges with an end in V_j , so that there are at most $2t\Delta - 1$ other events A_g which are dependent on A_f . We set $d = 2t\Delta - 1$ so that in the dependency graph for events $\{A_f \mid f \in E\}$ has maximum degree at most $d = 2t\Delta - 1$. Also,

$$\begin{aligned} P(A_f) &= \frac{1}{t^2} = \frac{1}{t \lceil 2e\Delta \rceil} \leq \frac{1}{t(2e\Delta)} \\ &= \frac{1}{e(2t\Delta)} = \frac{1}{e((2t\Delta - 1) + 1)} = \frac{1}{e(d + 1)}. \end{aligned}$$

Lemma 13.18 (continued 2)

Lemma 13.18. Let $G = (V, E)$ be a simple graph and let $\{V_1, V_2, \dots, V_k\}$ be a partition of V into k sets, each of cardinality at least $2e\Delta$ (again, “ e ” here is the base of the natural log function). Then there is a stable set S in G such that $|S \cap V_i| = 1$ for $1 \leq i \leq k$.

Proof (continued). So the hypotheses of Theorem 13.14 hold and hence $P(\cap_{f \in E} \bar{A}_f) > 0$. So by Note 13.5.A, there exists a set S which (by construction) intersects each V_i in one point and each edge f of G has its ends in different sets V_i and V_j (since \bar{A}_f holds; i.e., f does not have both ends in the same V_i). Since S contains exactly one point from each V_i , then every edge incident to a vertex in S has its other end outside of S . That is, S is a stable set, as claimed. \square

Theorem 13.19

Theorem 13.19. Let $G = (V, E)$ be a simple $2r$ -regular graph with girth at least $2e(4r - 2)$ (again, “ e ” here is the base of the natural log function). Then $\text{la}(G) = r + 1$.

Proof. We saw in Note 13.5.B that $\text{la}(G) \leq r + 1$. We now borrow a result from Section 16.4 (“Perfect Matchings and Factors”): “Every $2r$ -regular graph admits a decomposition into 2-factors” (this is Exercise 16.4.16).

Consider such a decomposition $\{F_1, F_2, \dots, F_r\}$ of G and let C_i , for $1 \leq i \leq k$, be the constituent cycles of these 2-factors (since a 2-factor is a 2-regular graph, then it is a vertex disjoint union of cycles). Define the edge sets $V_i = E(C_i)$ for $1 \leq i \leq k$. The line graph H of G is $(4r - 2)$ -regular (since each edge of G is adjacent to $2r - 1$ other edges at each of its ends).

Theorem 13.19 (continued)

Theorem 13.19. Let $G = (V, E)$ be a simple $2r$ -regular graph with girth at least $2e(4r - 2)$ (again, “ e ” here is the base of the natural log function). Then $\text{la}(G) = r + 1$.

Proof (continued). Because G has girth at least $2e(4r - 2)$ by hypothesis, then the edge sets $\{V_1, V_2, \dots, V_k\}$ partition the vertex set $V(H)$ into k sets, each of cardinality at least $2e(4r - 2)$ (since each is the set of edges of a cycle in G). So the hypotheses of Lemma 13.18 are satisfied by the line graph H , so H has a stable set S meeting each set V_i in one vertex. Define the subgraphs $L_i = F_i \setminus S$ for $1 \leq i \leq r$. Since F_i is a collection of vertex disjoint cycles in G and S includes an edge of each of these cycles, then each L_i is a linear forest on G . Also, if we set L_0 equal to the subgraph of G which has edge set S and vertex set of all ends of edges in S (so that L_0 is a linear forest where each tree has one edge because S is a stable set). Then $\{L_0, L_1, \dots, L_r\}$ is a decomposition of G into $r + 1$ linear forests. So $\text{la}(G) = r + 1$. \square