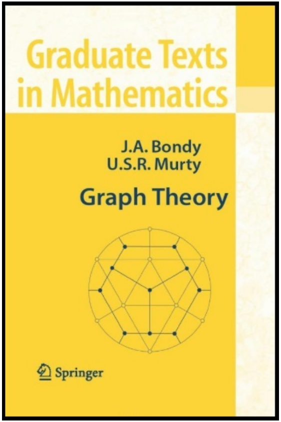
Graph Theory

Chapter 13. The Probabilistic Method

13.5. The Local Lemma—Proofs of Theorems



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Theorem 13.12

Theorem 13.12. The Local Lemma.
Let A_i, where $i \in N$, be events in a finite probability space (Ω, P) and let $N_i \subseteq N$ where $i \in N$. Suppose that, for all $i \in N$,

(i) A_i is independent of the set of events $\{A_j \mid J \in N_i\}$,

(ii) for each $i \in N$, there is a constant p_i where $0 < p_i < 1$, and for each $i \in N$ we have $P(A_i) = p_i \prod_{j \in N_i}(1 - p_j)$.

Set $B_i = \overline{A}_i$ where $i \in N$. Then, for any two disjoint subsets $R, S \subseteq N$,

$$P(B_R \cap B_S) \geq P(B_R) \prod_{i \in S}(1 - p_i). \qquad (13.15)$$

In particular, when $R = \varnothing$ and $S = N$,

$$P\left(\cap_{i \in N} \overline{A}_i\right) \geq \prod_{i \in N}(1 - p_i) > 0. \qquad (13.16)$$

Theorem 13.12 (continued 1)

Proof. If $S = \varnothing$ then $B_S = \cap_{i \in S} B_i = \cap_{i \in S} \overline{A}_i = \Omega$ (we could take this as the intersection of no sets) and $\prod_{i \in S}(1 - p_i) = 1$ (similarly, this could be taken as the definition of a product of no numbers), so

$$P(B_R \cap B_S) = P(B_R \cap \Omega) = P(B_R) = P(B_R)(1) \geq P(B_R) \prod_{i \in S}(1 - p_i),$$

and equation (13.15) holds when $S = \varnothing$.

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and equation (13.15) holds when $S = \varnothing$.

If $|S| = 1$ and $S = \{i\}$, then $B_S = B_i$ and $\prod_{j \in S}(1 - p_j) = 1 - p_i$. Define $R_1 = R \setminus N_i$ and $S_1 = R \cap N_i$ (so that $R = R_1 \cup S_1$). Then

$$\begin{aligned} P(A_i \cap B_R) &\leq P(A_i \cap B_{R_1}) \text{ since } R_1 \subseteq R \text{ and so} \\ &\quad B_R = \cap_{i \in R} \overline{A}_i \subseteq \cap_{i \in R_1} \overline{A}_i = B_{R_1} \text{ and } A_i \cap B_R \subseteq A_i \cap B_{R_1} \\ &= P(A_i)P(B_{R_1}) \text{ since } A_i \text{ is independent of} \\ &\quad \{A_j \mid j \notin N_i\} \supseteq R_1 \text{ by hypothesis (i).} \qquad (*) \end{aligned}$$

Theorem 13.12 (continued 1)

Proof. If $S = \varnothing$ then $B_S = \cap_{i \in S} B_i = \cap_{i \in S} \overline{A}_i = \Omega$ (we could take this as the intersection of no sets) and $\prod_{i \in S}(1 - p_i) = 1$ (similarly, this could be taken as the definition of a product of no numbers), so

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If $|S| = 1$ and $S = \{i\}$, then $B_S = B_i$ and $\prod_{j \in S}(1 - p_j) = 1 - p_i$. Define $R_1 = R \setminus N_i$ and $S_1 = R \cap N_i$ (so that $R = R_1 \mathbin{\dot\cup} S_1$). Then

$$\begin{aligned} P(A_i \cap B_R) &\leq P(A_i \cap B_{R_1}) \text{ since } R_1 \subseteq R \text{ and so} \\ & B_R = \cap_{i \in R} \overline{A}_i \subseteq \cap_{i \in R_1} \overline{A}_i = B_{R_1} \text{ and } A_i \cap B_R \subseteq A_i \cap B_{R_1} \\ &= P(A_i) P(B_{R_1}) \text{ since } A_i \text{ is independent of} \\ & \{A_j \mid j \notin N_i\} \supseteq R_1 \text{ by hypothesis (i).} \qquad (*) \end{aligned}$$

Theorem 13.12 (continued 2)

Proof (continued). Since $S_1 \subseteq N_i \subseteq N$, then by hypothesis (ii)

$$P(A_i) \le p_i \prod_{j \in N_i} (1 - p_j) \le p_i \prod_{j \in S_1} (1 - p_j) \qquad (**)$$

for some p_i with $0 < p_i < 1$ and some p_j with $0 < p_j < 1$. In Exercise 13.5.A(i) it is to be shown by induction on $|S_1|$ that

$$P(B_{R_1}) \prod_{j \in S_1} (1 - p_j) \le P(B_{R_1} \cap B_{S_1}). \qquad (\dagger)$$

Therefore

$$\begin{aligned} P(A_i \cap B_R) &\le P(A_i)P(B_{R_1}) \text{ by } (*) \\ &\le P(B_{R_1}) p_i \prod_{j \in S_1} (1 - p_j) \text{ by } (**) \\ &\le p_i P(B_{R_1} \cap B_{S_1}) \text{ by } (\dagger), \ldots \qquad (\dagger\dagger) \end{aligned}$$

Theorem 13.12 (continued 2)

Proof (continued). Since $S_1 \subseteq N_i \subseteq N$, then by hypothesis (ii)

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Therefore

$$\begin{aligned} P(A_i \cap B_R) &\leq P(A_i)P(B_{R_1}) \text{ by } (*) \\ &\leq P(B_{R_1}) p_i \prod_{j \in S_1} (1 - p_j) \text{ by } (**) \\ &\leq p_i P(B_{R_1} \cap B_{S_1}) \text{ by } (\dagger), \ldots \qquad (\dagger\dagger) \end{aligned}$$

Theorem 13.12 (continued 3)

Proof (continued). and so

$$\begin{aligned} P(B_R \cap B_S) &= P(B_R \cap B_i) \text{ since } S = \{i\} \\ &= P(B_R) - P(B_R \cap A_i) \text{ since} \\ &\quad B_R = (B_R \cap A_i) \mathbin{\dot\cup} (B_R \cap \overline{A}_i) = (B_R \cap A_i) \mathbin{\dot\cup} B_R \cap B_i) \\ &\geq P(B_R - p_i P(B_R) \text{ by } (\dagger\dagger) \\ &= P(B_R)(1 - p_i) = P(B_R) \prod_{i \in S} (1 - p_i), \end{aligned}$$

and equation (13.15) holds when $|S| = 1$.

If $|S| \geq 2$, then let R_1 and S_1 be nonempty disjoint sets which partition S so that $S = R_1 \mathbin{\dot\cup} S_1$. Then

$$\begin{aligned} P(B_R \cap B_S) &= P(B_R \cap B_{R_1 \cup S_1}) \\ &= P(B_R \cap B_{R_1} \cap B_{S_1}) \text{ by definition of } B\text{'s as intersections} \\ &= P(B_{R \cup R_1} \cap B_{S_1}) \text{ similarly.} \end{aligned}$$

Theorem 13.12 (continued 3)

Proof (continued). and so

$$\begin{aligned} P(B_R \cap B_S) &= P(B_R \cap B_i) \text{ since } S = \{i\} \\ &= P(B_R) - P(B_R \cap A_i) \text{ since} \\ &\quad B_R = (B_R \cap A_i) \mathbin{\dot\cup} (B_R \cap \overline{A}_i) = (B_R \cap A_i) \mathbin{\dot\cup} B_R \cap B_i) \\ &\geq P(B_R - p_i P(B_R) \text{ by } (\dagger\dagger) \\ &= P(B_R)(1 - p_i) = P(B_R) \prod_{i \in S} (1 - p_i), \end{aligned}$$

and equation (13.15) holds when $|S| = 1$.

If $|S| \geq 2$, then let R_1 and S_1 be nonempty disjoint sets which partition S so that $S = R_1 \mathbin{\dot\cup} S_1$. Then

$$\begin{aligned} P(B_R \cap B_S) &= P(B_R \cap B_{R_1 \cup S_1}) \\ &= P(B_R \cap B_{R_1} \cap B_{S_1}) \text{ by definition of } B\text{'s as intersections} \\ &= P(B_{R \cup R_1} \cap B_{S_1}) \text{ similarly.} \end{aligned}$$

Theorem 13.12 (continued 4)

Proof (continued). In Exercise 13.5.A(ii) it is to be shown by induction on $|S_1|$ that

$$\begin{aligned} P(B_{R\cup R_1} \cap B_{S_1}) &\geq P(B_{R\cup R_1}) \prod_{i\in S_1}(1-p_i) \\ &= P(B_R \cap B_{R_1}) \prod_{i\in S_1}(1-p)i). \qquad (\ddagger) \end{aligned}$$

In Exercise 13.5.A(iii) it is to be shown by induction on $|R \cup R_1|$ that

$$P(B_R \cap B_{R_1}) \geq P(B_R) \prod_{i\in R_1}(1-p_i). \qquad (\ddagger\ddagger)$$

Theorem 13.12 (continued 5)

Proof (continued).
Therefore

$$\begin{aligned} P(B_R \cap B_S) &= P(B_{R \cup R_1} \cap B_{S_1}) \\ &\geq P(B_R \cap B_{R_1}) \prod_{i \in S_1} (1 - p_i) \text{ by } (\ddagger) \\ &\geq P(B_R) \prod_{i \in R_1} (1 - p_i) \prod_{i \in S_1} (1 - p_i) \text{ by } (\ddagger\ddagger) \\ &= P(B_R) \prod_{i \in S} (1 - p_i) \text{ since } S = R_1 \cup S_1, \end{aligned}$$

so equation (13.15) holds when $|S| \geq 2$ and hence holds for all S and R subsets of N, as claimed. □

Theorem 13.14

Theorem 13.14. The Local Lemma—Symmetric Version.
let A_i, where $i \in N$, be events in a finite probability space (Ω, P) having a dependency graph with maximum degree d. Suppose $P(A_i) < 1/(e(d+1))$ for all $i \in N$ (where "e" here is the base of the natural log function). Then $P(\cap_{i \in N} \overline{A}_i) > 0$.

Proof. Set $p_1 = 1/(d+1) = p$ for $i \in N$ (this value of p maximizes the function $f(p) = p(1-p)^d$ for $p \in (0,1)$ and will give us a "uniform bound" on $p(A_i)$ in hypothesis (ii) of The Local Lemma). Now the sets N_i are defined from the dependency graph (N_i includes all neighbors of vertex i in the dependency graph, so we have event A_i is independent of the events $\{A_j \mid j \not\in N_i\}$, as required by hypothesis (i) of The Local Lemma).

Theorem 13.14

Theorem 13.14. THE LOCAL LEMMA—SYMMETRIC VERSION.
let A_i, where $i \in N$, be events in a finite probability space (Ω, P) having a dependency graph with maximum degree d. Suppose $P(A_i) < 1/(e(d+1))$ for all $i \in N$ (where "e" here is the base of the natural log function). Then $P(\cap_{i \in N} \overline{A}_i) > 0$.

Proof. Set $p_1 = 1/(d+1) = p$ for $i \in N$ (this value of p maximizes the function $f(p) = p(1-p)^d$ for $p \in (0,1)$ and will give us a "uniform bound" on $p(A_i)$ in hypothesis (ii) of The Local Lemma). Now the sets N_i are defined from the dependency graph (N_i includes all neighbors of vertex i in the dependency graph, so we have event A_i is independent of the events $\{A_j \mid j \notin N_i\}$, as required by hypothesis (i) of The Local Lemma).

Theorem 13.14 (continued)

Proof (continued). Now

$$\begin{aligned} P(A_i) &\leq \frac{1}{e(d+1)} \text{ by hypothesis} \\ &\leq \left(\frac{d}{d+1}\right)^d \left(\frac{1}{d+1}\right) \text{ since } 1+\frac{1}{d} \leq e^{1/d} \text{ by Exercise 13.2.1(b)} \\ &\qquad \text{with } x = 1/d, \text{ or } (1+1/d)^d \leq e \text{ or } \left(\frac{d+1}{d}\right)^d \leq e \\ &\qquad \text{or } \left(\frac{d}{d+1}\right)^d \geq \frac{1}{e} \\ &= p \prod_{j \in N_i} (1-p)^d \text{ since } p_i = p = \frac{1}{d+1} \text{ for all } i \in N. \end{aligned}$$

So hypothesis (ii) of the Local Lemma holds. Hence, by the Local Lemma (Theorem 13.12, the "in particular" part), $P(\cap_{i \in N} \overline{A}_i) > 0$, as claimed. □

Theorem 13.15

Theorem 13.15. Let $H = (V, \mathcal{F})$ be a hypergraph in which each edge has at least k elements and meets at most d other edges. If $e(d+1) \leq 2^{k-1}$ (again, "e" here is the base of the natural log function), then H is 2-colourable.

Proof. Consider a random 2-colouring of V, where each vertex receives one of two colours with probability $1/2$. For each edge F of H, denote by A_F the event that F is monochomatic. Then events A_F and A_G are independent unless edges F and G share vertices. Since an edge of H meets at most d other edges, then the dependence graph for the events $\{A_F \mid F \in \mathcal{F}\}$ has maximum degree d.

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Corollary 13.16

Corollary 13.16. Let $H = (V, \mathcal{F})$ be a k-uniform k-regular hypergraph, where $k \geq 9$. Then H is 2-colourable.

Proof. Since H is k-uniform, then each edge contains exactly k elements (i.e., vertices), and since H is k-regular then each vertex of H lies on k edges. So for given edge F of H, F contains k vertices and each lies on $k-1$ edges (along with edge F), so that edge F meets at most $d = k(k-1)$ other edges.

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Theorem 13.17

Theorem 13.17. Let D be a strict (i.e., "simple") k-diregular digraph where $k \geq 8$. Then D contains a directed even cycle.

Proof. Consider a random 2-colouring of V, where each vertex receives one of the two colours with probability $1/2$. For each vertex v of D, denote by A_v the event that $c(u) = c(v)$ for all $u \in N^+(v)$ (that is, A_v denotes the event that all outneighbors of v are the same colour as v). So A_v is independent of all A_u such that $(\{u\} \cup N^+(u)) \cap N^+(v) = \varnothing$ (that is, the outneighbors of v do not include any outneighbors of u nor u itself). Then A_v is dependent on some A_u when onr (or more) of the k outneighbors of v is one of the k outneighbors of u or u itself.

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Theorem 13.17. Let D be a strict (i.e., "simple") k-diregular digraph where $k \geq 8$. Then D contains a directed even cycle.

Proof (continued). Next, notice that $\overline{A}_v$ is the event that v has an outneighbor of a different colour. Since v has k outneighbors then $P(A_v) = 1/2^k$. With $d = k^2$ we have $P(A_i) = \dfrac{1}{2^k} \leq \dfrac{1}{e(d+1)} = \dfrac{1}{e(k^2+1)}$ for $k \geq 8$, since (1) for $k = 8$ we have $\frac{1}{2^k} = \frac{1}{2^8} = \frac{1}{256} \approx 0.0039$, $\frac{1}{e(k^2+1)} = \frac{1}{65e} \approx 0.0057$, and (2) $f(x) = \dfrac{1}{e(x^2+1)} - 2^{-x}$ has derivative $f'(x) = \dfrac{-2x}{e(x^2+1)^2} + (\ln 2)2^{-x} > 0$ for $k \geq 8$ so that f is increasing for $x \geq 8$ and hence $f(x) \geq f(8) > 0$ for all $x \geq 8$. So the hypotheses of Theorem 3.14 are satisfied and hence $P(\cap_{v \in V} \overline{A}_v) > 0$. That is, there is a 2-colouring of V such that every $v \in V$ has an outneighbor of the opposite colour.

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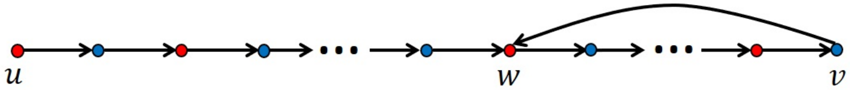
Theorem 13.17 (continued 2)

Theorem 13.17. Let D be a strict (i.e., "simple") k-diregular digraph where $k \geq 8$. Then D contains a directed even cycle.

Proof (continued). With respect to this colouring, let uPv be a maximal (length) properly 2-coloured directed path in D and let w be an outneighbor of v of the opposite colour of v. Since uPv is maximal, then w must be some vertex of uPv (or else $uPvw$ would be a longer properly 2-coloured path in D). So take the segment of P from w to v (denoted wPv) and then add the arc from v to w to produce a cycle in D. This cycle is properly coloured and so is an even length cycle, as claimed. □



Lemma 13.18

Lemma 13.18. Let $G = (V, E)$ be a simple graph and let $\{V_1, V_2, \ldots, V_k\}$ be a partition of V into k sets, each of cardinality at least $2e\Delta$ (again, "e" here is the base of the natural log function). Then there is a stable set S in G such that $|S \cap V_i| = 1$ for $1 \leq i \leq k$.

Proof. By deleting vertices from G if necessary, we may assume that $|V_i| = t = \lceil 2e\Delta \rceil$ for $1 \leq i \leq k$ (we'll show the existence of a stable set S under these conditions, then the deleted vertices and relevant edges can be added back in to G and this won't have an effect on the stability of set S nor on the intersection property of set S). We select one vertex v_i at random from each set V_i for $1 \leq i \leq k$, and set $S = \{v_1, v_2, \ldots, v_n\}$.

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For an edge f of G, let A_f denote the event that both ends of f belong to S. Since $|V_i| = t$ for each i then $P(A_f) = 1/t^2$ for all $f \in E$. In Exercise 13.5.A is to be shown that A_f is dependent only on those events A_g such that an end of g lies in the same set V_i as an end of f.

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Lemma 13.18 (continued 1)

Lemma 13.18. Let $G = (V, E)$ be a simple graph and let $\{V_1, V_2, \ldots, V_k\}$ be a partition of V into k sets, each of cardinality at least $2e\Delta$ (again, "e" here is the base of the natural log function). Then there is a stable set S in G such that $|S \cap V_i| = 1$ for $1 \leq i \leq k$.

Proof (continued). For a given A_f where f has its ends in V_i and V_j (where possibly $i = j$), there are at most $t\Delta$ edges with an end in V_i and at most $t\Delta$ edges with an end in V_j, so that there are at most $2t\Delta - 1$ *other* events A_g which are dependent on A_f. We set $d = 2t\Delta - 1$ so that in the dependency graph for events $\{A_f \mid f \in E\}$ has maximum degree at most $d = 2t\Delta - 1$. Also,

$$P(A_f) = \frac{1}{t^2} = \frac{1}{t\lceil 2e\Delta \rceil} \leq \frac{1}{t(2e\Delta)}$$

$$= \frac{1}{e(2t\Delta)} = \frac{1}{e((2t\Delta - 1) + 1)} = \frac{1}{e(d+1)}.$$

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Lemma 13.18 (continued 2)

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Proof (continued). So the hypotheses of Theorem 13.14 hold and hence $P(\cap_{f \in E} \overline{A}_f) > 0$. So by Note 13.5.A, there exists a set S which (by construction) intersects each V_i in one point and each edge f of G has its ends in different sets V_i and V_j (since $\overline{A}_f$ holds; i.e., f does not have both ends in the same V_i). Since S contains exactly one point from each V_i, then every edge incident to a vertex in S has its other end outside of S. That is, S is a stable set, as claimed. □

Theorem 13.19

Theorem 13.19. Let $G = (V, E)$ be a simple $2r$-regular graph with girth at least $2e(4r - 2)$ (again, "e" here is the base of the natural log function). Then $\text{la}(G) = r + 1$.

Proof. We saw in Note 13.5.B that $\text{la}(G) \leq r + 1$. We now borrow a result from Section 16.4 ("Perfect Matchings and Factors"): "Every $2r$-regular graph admits a decomposition into 2-factors" (this is Exercise 16.4.16).

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Consider such a decomposition $\{F_1, F_2, \ldots, F_r\}$ of G and let C_i, for $1 \leq i \leq k$, be the constituent cycles of these 2-factors (since a 2-factor is a 2-regular graph, then it is a vertex disjoint union of cycles). Define the *edge* sets $V_i = E(C_i)$ for $1 \leq i \leq k$. The line graph H of G is $(4r - 2)$-regular (since each edge of G is adjacent to $2r - 1$ other edges at each of its ends).

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Proof (continued). Because G has girth at least $2e(4r-2)$ by hypothesis, then the edge sets $\{V_1, V_2, \ldots, V_k\}$ partition the *vertex* set $V(H)$ into k sets, each of cardinality at least $2e(4r-2)$ (since each is the set of edges of a cycle in G). So the hypotheses of Lemma 13.18 are satisfied by the line graph H, so H has a stable set S meeting each set V_i in one vertex. Define the subgraphs $L_i = F_i \setminus S$ for $1 \leq i \leq r$. Since F_i is a collection of vertex disjoint cycles in G and S includes an edge of each of these cycles, then each L_i is a linear forest on G. Also, if we set L_0 equal to the subgraph of G which has edge set S and vertex set of all ends of edges in S (so that L_0 is a linear forest where each tree has one edge because S is a stable set). Then $\{L_0, L_1, \ldots, L_r\}$ is a decomposition of G into $r+1$ linear forests. So $\text{la}(G) = r + 1$. □

Theorem 13.19 (continued)

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